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Solid spaces and absolute retracts

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1. The well-known Tietze extension theorem says that a bounded real-valued continuous function defined on a closed subset of a normal space can be extended to a function defined on the whole space and there having the same lower and upper bounds as the original function. This theorem, which is of great importance in the theory of normal spaces, can also be looked upon as giving a property of the closed interval: Any mapping into a closed interval of a closed subset of a normal space has an extension to the whole space. Steenrod [7] has suggested the name "solid" for spaces having this property.

Definition. A space X is called *solid*, if for any normal space Y, any closed subset B of Y, and any mapping $f:B \to X$ there exists an extension $F: Y \to X$ of f.

Tietze's extension theorem then simply asserts that a closed interval is solid.

Lemma 1.1. *Any topological product of solid spaces is solid.*

Proof. For if X is the topological product of the spaces X_α, then a mapping $f:B \to X$ of a closed subset B of a normal space Y is equivalent to a collection of mappings $f_\alpha:B \to X_\alpha$, obtained from f by projection onto each X_α. X_α being solid, f_α can be extended to Y. These extensions together define an extension of f.

Since a closed interval is a solid space, so also is any cube, i.e. a product of closed intervals. In particular the Hilbert cube is solid.

2. There is a strong connection between the concept of a solid space and of an absolute retract. We shall in this paper study this connection.

Using Kuratowski's extension ([5] p. 270) of Borsuk's original definition, we mean by an absolute retract (abbreviated AR) a separable metric space X such that, whenever X is imbedded as a closed subset of a separable metric space Z, X is a retract of Z.

Similarly we mean by an absolute neighborhood retract (abbreviated ANR) a separable metric space X such that, whenever X is imbedded as a closed subset of a separable metric space Z, X is a retract of some neighborhood of X in Z.

Lemma 2.1. *A retract of a solid space is solid.*

Proof. Assume X is a retract of Z. Denote the retraction by $r:Z \to X$. Let $f:B \to X$ be a given mapping of a closed subset B of a normal space Y. Considering f as a mapping into Z we have an extension $F: Y \to Z$. Then $rF: Y \to X$ is an extension of f to Y relative to X.