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Estimates of harmonic measures

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Introduction

Estimates of harmonic measures in terms of Euclidean quantities are useful in many situations. In the two-dimensional case one can apply methods of conformal mapping and extremal lengths, and many sharp results are well known. Different means of harmonic measures can be studied in the n-dimensional case. This paper is intended to provide a survey of methods available to estimate harmonic measures.

The two-dimensional case is treated in Chapter I. The second paragraph contains well-known distortion inequalities from the theory of conformal mapping, and § 3 contains well-known results from the theory of extremal lengths. In § 4 we prove two symmetrization theorems with the aid of a result from § 3. In § 5 we apply results from § 3 to comb domains.

Chapter II gives n-dimensional methods. In § 6 a method of Carleman [6.1] is applied to harmonic measures. The derivation of Carleman's method in Theorem 6.1 follows that of Dinghas [6.3]. The estimates of harmonic measures in Theorems 6.2 and 6.3 are new in the case $n>2$. In § 7 we treat Nevanlinna's mean value in a special case. In § 8 we prove some symmetrization results with probabilistic methods.

The main problem is to provide upper bounds for harmonic measures. Lower bounds are discussed in § 2 and § 7.

Bearing in mind the possibility of exhausting a given domain with more regular domains we have not aimed at generality in assumptions about the domains considered.

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1. *Definitions*

R^n is the n-dimensional Euclidean space, $n \geqslant 2$, with points $z=(x_1, y_1, \dots, y_{n-1})=(x_1, y)$. In Chapter I we treat the case $n=2$ and prefer to write $z=x+iy$. The following definitions are then to be understood with $\operatorname{Re} z$ instead of x_1.

D denotes a domain (open connected set) and ∂D the boundary of D.

$\Theta_x=\{z \mid x_1=x,\ z \in D\}$.

D_x is the subdomain of $\{z \mid x_1<x,\ z \in D\}$ that contains a given point z_0.

$\theta_x=\{z \mid z \in \Theta_x,\ z \in \partial D_x\}$.

Without D being specified D_ξ denotes a domain in $\{z \mid x_1<\xi\}$ with part of its boundary on $\{z \mid x_1=\xi\}$ and θ_ξ then denotes the interior of $\{z \mid x_1=\xi,\ z \in \partial D_\xi\}$.