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Note on multiplicities of ideals

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Introduction

In this note we prove some formulae involving lengths and multiplicities of ideals. Our results are incomplete but, in the absence of more final results, they are perhaps not without interest.

We begin by proving a generalization of Samuel's well-known inequality $e(x_1, \ldots, x_r) \leqslant L(x_1, \ldots, x_r)$ interrelating the multiplicity and length of an ideal generated by a system of parameters in a local ring (Theorem 1). By combining this generalization with an argument in [2] we obtain an asymptotic expression for $e(x_1, \ldots, x_r)$ which is more general than the one given in the paper cited (Theorem 2).

The rest of the note is independent of the results just mentioned and mainly concerns flat couples of local rings (Serre, [10], pp. 34–41). Let (Q_0, Q) be such a couple with maximal ideals $(\mathfrak{m}_0, \mathfrak{m})$. Assume that $\mathfrak{m}_0 Q$ is a $\mathfrak{m}$-primary ideal or, equivalently, that Q_0 and Q have the same dimension. Denote by $e(Q_0)$ and $e(Q)$ the multiplicities of $\mathfrak{m}_0$ and $\mathfrak{m}$ respectively. We prove that if the dimension of Q_0 and Q is less than or equal to two, then

$$e(Q_0) \leqslant e(Q),$$

and we make some further observations in support of a conjecture that this inequality is always true. The truth of the conjecture would imply the general truth of the inequality

$$e(Q_{\mathfrak{p}}) \leqslant e(Q)$$

for prime ideals $\mathfrak{p}$ of Q satisfying $\dim \mathfrak{p} + \operatorname{rank} \mathfrak{p} = \dim Q$. For, according to Nagata ([4], § 13), this inequality is valid when Q is complete, and one could pass from Q to its completion Q^* by means of a suitable flat couple $(Q_{\mathfrak{p}}, Q^*_{\mathfrak{p}^*})$. — However, our arguments are powerless in the general case. The result for dimension two is obtained by using ideals similar to form ideals but generated by power products of the variables. An application of these ideals also gives another estimate which bears a slight resemblance to the formula $e(Q_{\mathfrak{p}}) \leqslant e(Q)$ (Theorem 3).

Serre defines flat couples in homological terms. In the present note we use hardly anything of the homological machinery, and in an appendix we give an alternative non-homological definition of flat couples, which would serve us equally well and which ties up this new concept with an older result of Samuel-Nagata.