Communicated 13 May 1959 by EINAR HILLE

Suboperative functions and semi-groups of operators

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1. Let S be a locally compact space endowed with a composition law $(s, t) \to s \circ t$ defined for $(s, t) \in D \subset S \times S$, $\mathfrak{J}$ the set of open parts of S, $\mathfrak{B} \subset \mathfrak{J}$ a filter basis on S and μ a positive Radon measure[1] on S. Suppose that:

1) $(s, t) \to s \circ t$ is continuous on D;

2) for each $t \in S$ and $W \in \mathfrak{V}(t)$ there exist $U \in \mathfrak{B}$ and $V \in \mathfrak{V}(t)$ such that $U \circ V \subset W$;

3) $\mu(U) \neq 0$ for every non void set $U \in \mathfrak{J}$.

A function f on S to $[-\infty, \infty)$ is said to be suboperative if $f(s \circ t) \leqslant f(s) + f(t)$ for every $(s, t) \in D$. In this paper various properties of suboperative functions and various measurability and continuity properties of certain representations of S are given. Functions satisfying conditions less restrictive than the suboperative ones are also studied.

Most of the results proved below may be considered as extensions of those given in (5) p. 92–94 and (6) p. 741–748. In fact, the subject matter of this paper has been suggested by the reading of (5) p. 92–94 and (6) p. 741–748.

2. It will be supposed in this paragraph that the following condition is satisfied (we denote with $\mathfrak{K}$ the set of all compact parts of S):

(H_1) *For every $t \in S$ there exist $K_t \in \mathfrak{K} \cap \mathfrak{V}(t)$, $V_t \in \mathfrak{B}$ and a continuous mapping $(s, u) \to p_s(u)$ of $K_t \times V_t$ into S having the following properties:* (i) $u \circ p_s(u) = s$ *for every $s \in K_t$ and $u \in V_t$;* (ii) $\lim_{\mathfrak{B}} p_s = s$ *for every $s \in K_t$;* (iii) *there is a constant $\lambda(K_t) > 0$ which satisfies the inequality $\mu(p_s^{-1}(U)) \leqslant \lambda(K_t)\,\mu(U)$ for every $s \in K_t$ and $U \in \mathfrak{J}$.*

The condition (H_1) is a refinement of condition (P_9^*), (6) p. 741. Let us remark that if (H_1) is satisfied, then, for every $s \in K_t$ and μ-measurable set $A \subset S$, the set $p_s^{-1}(A)$ is μ-measurable and $\mu(p_s^{-1}(A)) \leqslant \lambda(K_t)\,\mu(A)$.

Proposition 1.—*Let $t \in S$, $K_1 \subset V_t$, $K_1 \in \mathfrak{K}$ and f a locally μ-integrable mapping of S into a Banach space E. Then for every $t' \in K_t$*

4) $$\lim_{s \to t', s \in K_t} \int \varphi_{K_1}(u)\, \| f(p_s(u)) - f(p_{t'}(u)) \|\, d\mu(u) = 0\,[2].$$

[1] Various indications concerning the terminology are given in paragraph 6.

[2] We take $v(u) = 0$ if $v = h \circ p_s$ and $u \notin V_t$.