ARKIV FÖR MATEMATIK Band 4 nr 5

Communicated 9 April 1958 by HARALD CRAMÉR and LENNART CARLESON

# Likelihood ratios of Gaussian processes[1]

By T. S. PITCHER[2]

## 1. Introduction

Let $x(t)$, $A \leqslant t \leqslant B$ be a real Gaussian stochastic process with autocorrelation function $R(s,t)$. Each choice of a mean value function $f(t)$ for the process establishes a measure $m_f$ on the set of sample functions made into a measure space in the usual way [1]. In statistical applications one often wishes to know when $m_f$ and $m_g$ are totally singular and when they are absolutely continuous with respect to each other, i.e., when the likelihood ratio exists. In the latter case it is desirable to be able to compute $(d\,m_f/d\,m_g)(x)$ in terms of the sample function $x(t)$.

The transformation on the space of sample functions which carries $x(t)$ into $x(t)+f(t)$ preserves measurability and carries $m_g$ into $m_{f+g}$, i.e., if $a(x)$ is a measurable function so is $a(x+f)$ and we have

$$\int a(x)\,d\,m_{f+g} = \int a(x+f)\,d\,m_g.$$

The following lemma shows that it is sufficient to consider the case $g=0$.

**Lemma 1.1.** $m_f$ and $m_g$ are totally singular if and only if $m_{f-g}$ and $m_0$ are. $m_f$ is absolutely continuous with respect to $m_g$ if and only if $m_{f-g}$ is absolutely continuous with respect to $m_0$ and in this case $(d\,m_f/d\,m_g)(x) = (d\,m_{f-g}/d\,m_0)\,(x-g)$.

*Proof.* If $m_f(A)=1$ and $m_g(A)=0$ then $m_{f-g}(A+g)=1$ and $m_0(A+g)=0$ which proves the first assertion. If $d\,m_f/d\,m_g$ exists then

$$\int a(x)\,\frac{dm_f}{dm_g}\,(x+g)\,dm = \int a(x-g)\,\frac{dm_f}{dm_g}\,(x)\,dm_g = \int a(x-g)\,dm_f = \int a(x)\,dm_{f-g}$$

which proves the second assertion.

From now on we shall assume that $R$ is continuous and bounded, and that the process is separable. We will write $m$ for $m_0$, $m_f \| m$ if $m_f$ and $m$ are totally singular, and $m_f \equiv m$ if $m_f$ and $m$ are mutually absolutely continuous.

**Lemma 1.2.** If $A$ and $B$ are finite $x(t)$ is in $L_2$ with $m$ probability one. If $f$ is not in $L_2$, $m_f \| m$.

[1] The research in this paper was supported jointly by the U.S. Army, U.S. Navy, and U.S. Air Force under contract with the Massachusetts Institute of Technology.

[2] Staff Member, Lincoln Laboratory, Massachusetts Institute of Technology.