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The Frobenius–Nirenberg theorem

By LARS HÖRMANDER

The extension of the classical Frobenius theorem given by Nirenberg [5], combined with the Poincaré lemma for the operators d and $\bar{\partial}$, gives conditions on a first order system of differential equations

$$P_j u = f_j, \; j = 1, \dots, N, \tag{1}$$

for one unknown, which guarantee the existence of local solutions when $f = (f_1, \dots, f_N)$ satisfies the obvious integrability conditions. In fact, using the classical Frobenius theorem and the theorem of Newlander and Nirenberg [4], Nirenberg determined when it is possible to reduce (1) by a change of variables to a system of equations where each P_j is either $\partial/\partial x^j$ or $\partial/\partial x^j + i\,\partial/\partial x^{j+1}$ for some j. Now Kohn [3] has given a proof of the Newlander–Nirenberg theorem which is based on L^2 estimates. We shall show here that a modified form of his approach leads to a direct proof of existence theorems for the system of equations (1). These results are global and they require only very light smoothness assumptions on the coefficients.

Le P_j, $j = 1, \dots, N$, be first order differential operators in an open set $\Omega \subset \mathbf{R}^n$,

$$P_j = \sum_{k=1}^{n} a_j^k \,\partial/\partial x_k + a_j^0, \; j = 1, \dots, N.$$

(No additional difficulty arises if Ω is a manifold and instead of the operators $(P_1, \dots, P_N)$ we have a first order differential operator P between two complex vector bundles over Ω, with fibers of dimension 1 and N respectively. However, this more general framework would make the notations somewhat heavier.) We denote the principal parts by p_j,

$$p_j = \sum_{k=1}^{n} a_j^k \,\partial/\partial x_k,$$

and we write $\bar{p}_j$ for the operator obtained by complex conjugation of the coefficients.

Since equations of the form (1) allow us to obtain additional first order equations by forming brackets,

$$[P_j, P_k]u = (P_j P_k - P_k P_j)\,u = P_j f_k - P_k f_j,$$