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On absolute convergence of Fourier series

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§ 1. Let f be an integrable function on $(0, 2\pi)$ and periodic with period 2π, and its Fourier series be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

If $\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty$, then we say that the Fourier series of f converges absolutely and we write $f \in A$. If $f \in A$, then f must be bounded and continuous.

We define the modulus of continuity of f by

$$\omega(\delta; f) = \sup_{|x-x'| \leqslant \delta} |f(x) - f(x')|$$

and the integrated modulus of continuity of f by

$$\omega_p(\delta; f) = \sup_{0<h\leqslant\delta} \left(\int_0^{2\pi} |f(x+h) - f(x)|^p \, dx \right)^{1/p}$$

for $p \geqslant 1$. It is known that $\omega_p(\delta; f) \leqslant \omega(\delta; f)$ and $\lim_{p\to\infty} \omega_p(\delta; f) = \omega(\delta; f)$.

Concerning absolute convergence of Fourier series there are two famous theorems, one is due to S. Bernstein and the other to A. Zygmund.

Bernstein's theorem ([1], p. 241; [2], p. 154) reads as follows:

Theorem I. *If*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \omega\left(\frac{1}{n}; f\right) < \infty,$$

then $f \in A$.

This was generalized hy O. Szász ([2], p. 155) in the following form:

Theorem II. *If*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \omega_2\left(\frac{1}{n}; f\right) < \infty,$$

then $f \in A$.

Zygmund's theorem ([1], p. 242; [2], p. 160)[1] reads as follows.

[1] The condition of this form was first formulated by E. Hille and J. D. Tamarkin [3].