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Pure submodules

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1. Introduction

The notion of purity is extremely important in abelian group theory. One reason for this is that there are enough pure-injective (and also enough pure-projective) groups, which makes it possible to apply the methods of relative homological algebra. When extending the purity concept to modules over arbitrary rings, it is therefore natural to look for two types of generalizations, one giving enough pure-projectives and one giving enough pure-injectives. The most useful notion of purity of the first type seems to be the one introduced by Butler and Horrocks [2]. We will give a rather detailed treatment of it in sec. 6–9. By dualization we obtain a theory of copurity, with enough copure-injectives (sec. 10). The usual examples of copurity in module categories are related to exactness properties of the tensor product (sec. 11 and 13).

C. L. Walker [15] has proposed another way of defining purity, which also generalizes the traditional notions of purity for modules over Dedekind rings, but which seems somewhat less natural for modules over more general rings (in particular it does not include purity in the sense of Cohn [4] and Bourbaki ([1], ch. 1, § 2, exercise 24).

Our theory of purity and copurity probably covers all reasonable notions of purity for abelian groups and modules that have been used in the literature. Most of the time we will work in an abelian category with a projective generator, thus being rather near to a category of modules. Actually, our examples (collected in sec. 9 and 13) deal only with modules.

We also study the theories of torsion and divisibility which are associated to the concepts of purity and copurity, and are "torsion theories" in the technical sense of Dickson [5].

The first five sections are of a preliminary nature. Sec. 2 and 3 contain some general remarks on proper classes, in sec. 4 Maranda's theory of pure-essential extensions is generalized, and sec. 5 contains some remarks on the relative homological algebra associated to a torsion theory, related to the results of Walker [15].

2. Proper classes

Let $\mathcal{A}$ be an abelian category. Consider a class $\mathcal{E}$ of short exact sequences of $\mathcal{A}$, such that every sequence isomorphic to a sequence in $\mathcal{E}$ also is in $\mathcal{E}$. The corresponding class of monomorphisms (epimorphisms) is written $\mathcal{E}_m(\mathcal{E}_e)$. $\mathcal{E}$ is called a *proper class* if it satisfies: