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Fractional categories

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Introduction

In this paper we study the "localization" of a category $\mathcal{C}$ with respect to certain subcategories $\mathcal{S}$. This is done by a category of "right fractions", $\mathcal{C}\mathcal{S}^{-1}$ and a functor ϕ: $\mathcal{C}\to\mathcal{C}\mathcal{S}^{-1}$. In § 1 conditions for the existence of $\mathcal{C}\mathcal{S}^{-1}$ are given and it turns out that ϕ is left exact.

In § 2 the existence of left adjoint $^*\phi$ of ϕ is discussed. If $^*\phi$ exists, a full subcategory $\check{\mathcal{C}}$ of $\mathcal{C}$ is defined. $\check{\mathcal{C}}$ consists of those objects A such that $^*\phi\cdot\phi(A)\approx A$. It follows that $\check{\mathcal{C}}$ is equivalent ot $\mathcal{C}\mathcal{S}^{-1}$.

If in the *dual* case (i.e. the right adjoint ϕ^* exists) ϕ is exact and $\mathcal{C}$ is a category of set-valued presheaves then $\check{\mathcal{C}}$ is a category of sheaves for some Grothendieck topology. Furthermore the imbedding functor $\check{\mathcal{C}}\to\mathcal{C}$ has a left adjoint which is the associated sheaf functor.

Section 3 is devoted to a study of the functional properties of $\mathcal{C}\mathcal{S}^{-1}$, i.e. its behaviour under functors and under change of $\mathcal{S}$.

In § 4 properties inherited from $\mathcal{C}$ to $\mathcal{C}\mathcal{S}^{-1}$ are studied under various conditions on $\mathcal{S}$. If ϕ^* exists, it follows that if $\mathcal{C}$ is abelian (a topos), then $\mathcal{C}\mathcal{S}^{-1}$ is also.

In § 5 some examples are given when $\mathcal{C}$ is a special abelian category, in particular when $\mathcal{C}$ is the category of modules over a commutative ring.

Notation and generalities

All categories in this paper are sets. Let $\mathcal{U}$ be a fixed universe (see [10], Exp. VI). If nothing else is stated a category $\mathcal{C}$ will mean a *$\mathcal{U}$-category*, i.e. for any pair of objects A, B in C there is a bijection from $\mathrm{Hom}_{\mathcal{C}}(A, B)$ onto a set belonging to $\mathcal{U}$. $\mathcal{E}ns$ is the category of sets of cardinality less than Card $(\mathcal{U})$. A category $\mathcal{C}$ is *small* if the set underlying $\mathcal{C}$ is in $\mathcal{E}ns$.

If $\mathcal{C}$ is a category we denote the set of objects of $\mathcal{C}$ by $\mathcal{C}_0$ and identify $\mathcal{C}_0$ with the identities of $\mathcal{C}$. If F: $\mathcal{C}\to\mathcal{D}$ is a functor, then a functor $^*F$: $\mathcal{D}\to\mathcal{C}$ is a *left adjoint* of F (and F is a *right adjoint* of $^*F$) if there is a functor isomorphism $\mathrm{Hom}_{\mathcal{C}}(^*F\cdot,\cdot)\approx \mathrm{Hom}_{\mathcal{D}}(\cdot, F\cdot)$. Let $\mathcal{J}$ be a small category and let C: $\mathcal{C}\to Hom\,(\mathcal{J}, \mathcal{C})$ denote the functor that maps each $A\in\mathcal{C}_0$ onto the constant functor C_A defined by $C_A(\alpha)=1_A$ for all $\alpha\in\mathcal{J}$. C maps the morphisms of $\mathcal{C}$ in the obvious way.

If C has a right adjoint, $C^*=\lim_{\leftarrow}$: $Hom\,(\mathcal{J}, \mathcal{C})\to\mathcal{C}$ then $\mathcal{C}$ is said to have $\mathcal{J}$-$\lim_{\leftarrow}$. In particular if $\mathcal{C}$ has $\mathcal{J}$-$\lim_{\leftarrow}$ for all small (finite) $\mathcal{J}$, then $\mathcal{C}$ has (finite) $\lim_{\leftarrow}$. $^*C=\lim_{\rightarrow}$ is defined dually.