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On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$

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1. Introduction

This papér treats various aspects of the partial differential equation

$$\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

This equation was derived in [1] where an extension problem was studied, and it turned out that (1) is closely connected to this extension problem (Theorems 6, 7 and 8 in [1]). The equation is quasi-linear and parabolic ($AC - B^2 = 0$), and is not of any classical type. The results from [1] will be used very little in this paper. As far as the author knows, the equation (1) has not been treated before, apart from the paper [1].

Let $u(x, y)$ be a solution of (1) and let C be a trajectory of the vector field grad u. Then it is proved in Section 2 that C is either a convex curve or a straight line, and this result, together with a formula for the curvature of C, is fundamental for the later sections.

In Section 3 we consider two particular classes of solutions to (1).

Section 4 is devoted to a discussion of the regularity of solutions to (1). It turns out that a solution for which the trajectories of grad u are convex curves, is infinitely differentiable.

In Section 5 we consider some differential-geometric aspects of (1).

Section 6 contains an estimate for $|\text{grad}\, u|$. A consequence of this estimate is that a nonconstant solution of (1) has no stationary points.

In Section 7 we consider solutions of (1) outside a compact set and solutions in the whole plane. The latter ones turn out to be linear functions only.

In Section 8 we consider the behaviour of $|\text{grad}\, u|$ near the boundary of a region.

Section 9, finally, contains a few results on the Dirichlet problem for (1).

In this paper, we will only consider classical solutions of (1), that is, solutions in C^2. We will not discuss extensions of the results to the case of more than two independent variables.

2. Some preliminary considerations

A lemma on the curvature of a streamline

We introduce the notation

$$A(\Phi) \equiv \Phi_x^2 \Phi_{xx} + 2\Phi_x \Phi_y \Phi_{xy} + \Phi_y^2 \Phi_{yy}.$$

It is easy to see that

$$A(\Phi) = \tfrac{1}{2}\, \text{grad}\, \{(\text{grad}\, \Phi)^2\} \cdot \text{grad}\, \Phi. \tag{2}$$