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On extensions of Lipschitz functions

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Let X be a compact metric space with metric d. A real-valued function f on X is Lipschitz if there exists a constant K such that $|f(x)-f(y)| \leqslant Kd(x, y)$ holds for all points $x, y \in X$. The class of functions Lipschitz on X is an algebra Lip (X, d). On Lip (X, d) we can introduce a norm in the following way. Let $f \in \text{Lip}\,(X, d)$. We put $\|f\|_c = \sup_{x\in X}|f(x)|$ and $\|f\|_d = \sup\,\{|f(x)-f(y)|/d(x, y)\,|\,x, y \in X \text{ and } x \neq y\}$. Finally we put $\|f\| = \|f\|_c + \|f\|_d$. It is easily seen that Lip (X, d) is a Banach algebra with this norm. A general investigation of Lip (X, d) is given in [1]. We shall freely use results and notations from that paper. In this paper our main result is the following: Let F be a closed subset of a compact metric space X with metric d. Let G be a closed set contained in F. Let $f \in \text{Lip}\,(F, d)$ be such that $\lim\,|f(x)-f(y)|/d(x, y) = 0$ as $d(x, G)$ and $d(y, G)$ tend to zero. Then there exists $H \in \text{Lip}\,(X, d)$ such that $H = f$ on F and $\lim\,|H(x)-H(y)|/d(x, y) = 0$ as $d(x, G)$ and $d(y, G)$ tend to zero. This result answers a question raised in [2] (see IV, Miscellaneous problems no. 8, p. 355). For the result above implies the following: Let X be a compact metric space with metric d. Let Lip (X, d) be the Banach algebra of functions Lipschitz on X. Let $I(F)$ be the ideal of functions vanishing on a closed set F contained in X. Let G be a closed subset of F. Let $J(G)$ be the smallest closed ideal with hull G. Let u be a continuous linear on $I(F)$ which vanishes on $J(G)$. Then u can be extended to a continuous linear form on $I(G)$ which vanishes on $J(G)$.

Remark. In [1] it is shown that $J(G)$ consists of all functions $f \in \text{Lip}\,(X, d)$ such that $f \in I(G)$ and $\lim\,|f(x)-f(y)|/d(x,y) = 0$ as $d(x, G)$ and $d(y, G)$ tend to zero.

In the proof of Theorem 1 we shall need the following result from [3].

Proposition 1. *Let X be a metric space with metric d. Let F be a closed subset of X. Let f be a bounded function Lipschitz on F, i.e. $\|f\|_c = \sup_{x\in X}|$ and $\|f\|_d = \sup\,\{|f(x)-f(y)|/d(x, y)\,|\,x, y \in F$ and $x \neq y\}$ are finite. Then there exists a function H on X such that $H = f$ on F and $H \in \text{Lip}\,(X, d)$ with $\|H\|_d = \|f\|_d$ and $\|H\|_c = \|f\|_c$.*

Proof. Let us put $H_1(x) = \sup_{y\in F}\{f(y) - \|f\|_d d(x, y)\}$. It is not hard to see that $H_1 = f$ on F and H_1 is Lipschitz on X with $\|H_1\|_d = \|f\|_d$. Now we only have to put $H(x) = H_1(x)$ if $|H_1(x)| \leqslant \|f\|_c$ and $H(x) = \|f\|_c$ if $H_1(x) > \|f\|_c$ and $H(x) = -\|f\|_c$ if $H_1(x) < -\|f\|_c$.

In the proof of Theorem 1 the following lemma will be useful.

Lemma 1. *Let F and G be two closed subsets of a metric space X with metric d, such that if $x \in F$ and $y \in G$ then there exists $z \in F \cap G$ with $4d(x, y) \geqslant d(x, z)$ and $4d(x, y) \geqslant d(y, z)$. Let $f \in \text{Lip}\,(F, d)$ and $g \in \text{Lip}\,(G, d)$ be such that $f = g$ on $F \cap G$. If we put $h = f$ on F and $h = g$ on G then $h \in \text{Lip}\,(F \cap G, d)$ and $\|h\|_d \leqslant 4(d\|f\|_d + \|g\|_d)$.*