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On the constant in Hölder's inequality

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With 5 figures in the text

It is well known that the '=' in the so-called Hölder's inequality in one of its forms

$$\sum a_n b_n \leq \left(\sum a_n^p\right)^{\frac{1}{p}} \left(\sum b_n^q\right)^{\frac{1}{q}} \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right) \tag{1 a}$$

or

$$\int_0^1 f(x)^r g(x)^{1-r} dx \leq \left(\int_0^1 f(x)\,dx\right)^r \left(\int_0^1 g(x)\,dx\right)^{1-r} \qquad (0<r<1) \tag{1 b}$$

occurs only if $a_n^p = A \cdot b_n^q$ for all n

or

$$f(x) \equiv g(x)$$

If $\{a_n\}$ and $\{b_n\}$ (or $f(x)$ and $g(x)$) are subjected to some restrictive conditions that exclude the proportionality just mentioned, then one can have only '<'.

We shall consider the most general form of Hölder's inequality, which includes (1 a) and (1 b), and study the value of the Lebesgue-Stieltjes integral

$$I_r = \int_E (f(x))^r (g(x))^{1-r} d\varphi(x) \qquad (0 \leq r \leq 1) \tag{1}$$

where $f(x)$ and $g(x)$ are non-negative functions and $\varphi(x)$ an increasing function on the set E. Hölder's inequality then takes the form

$$I_r \leq \theta_r \cdot I_1^r \cdot I_0^{1-r} \qquad (0 \leq \theta_r \leq 1) \tag{2}$$

In the following we shall assume throughout that the functions $f(x)$ and $g(x)$ are normalized in such a way that

$$I_1 = I_0 = 1$$

which is no real restriction of the study.