ARKIV FÖR MATEMATIK Band 3 nr 35

Read 13 February 1957

On linear recurrences with constant coefficients

By Trygve Nagell

1.—An arithmetical function $A(n)=A_n$ of n may be defined by *recursion* in the following way: The value of A_n is defined for $n=0, 1, 2, \ldots, m-1$, and there is given a rule indicating how the value of A_{m+n} may be determined when the values of A_μ are known for $\mu=n,\ n+1,\ n+2, \ldots,\ n+m-2,\ n+m-1$, n being an integer $\geqq 0$.

The infinite sequence

$$A_0, A_1, A_2, A_3, \ldots, A_n, \ldots$$

thus defined is said to be a *recurrent sequence*. We denote it by $\{A_n\}$. The rule of recursion has often the shape of a *recursive formula*.

For instance, the function $A_n=n!$ satisfies the recursive formula

$$A_{n+1}=(n+1)A_n \tag{1}$$

and the initial condition $A_0=1$.

The general solution of the recursive formula

$$A_{n+1}=2A_n \tag{2}$$

is obviously $$A_n=2^n A_0.$$

The arithmetical function A_n satisfying the recursive formula

$$A_{n+2}=\sqrt{A_{n+1}A_n} \tag{3}$$

is a function of n, A_0 and A_1.

Another example is the function A_n defined by the recursive formula

$$A_{n+2}=A_{n+1}+A_n \tag{4}$$

and the initial conditions $A_0=A_1=1$. In this case we get the following series:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots,$$

the so-called *Fibonacci numbers*.

2.—In Algebra and in Number Theory we often have to do with *linear recurrences*, that is to say recursive formulae of the type

$$A_{m+n}=a_1A_{m+n-1}+a_2A_{m+n-2}+\cdots+a_mA_n+b, \tag{5}$$