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A note on an inequality

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The following is a supplement to an earlier paper [9], where we have given a "triangular condition" which the exponents must fulfill in order that an inequality

$$\int_0^\infty x^\alpha |f(x)|^\beta dx \le K\left(\int_0^\infty x^{\alpha_1}|f|^{\beta_1}dx\right)^{\varkappa_1}\left(\int_0^\infty x^{\alpha_2}|f|^{\beta_2}dx\right)^{\varkappa_2} \tag{1}$$

should hold true. A number of authors have discussed the best possible value for K.

In this note we observe that the simple method we used in a special case in the cited note [9] gives—in the general case also—the extremal function and so the value of K.

By the transformations $x \to x^p$, $|f| \to x^q |f|^r$ we first bring (1) into the form

$$\int_0^\infty |f|\,dx \le K(\alpha, \beta_1, \beta_2)\left(\int_0^\infty x^\alpha |f|^{\beta_1}dx\right)^{\varkappa_1}\left(\int_0^\infty x^\alpha |f|^{\beta_2}dx\right)^{\varkappa_2}. \tag{2}$$

For brevity we here do not consider the simplest case $\beta_1 = \beta_2$. To satisfy the conditions in [9] we must have $\alpha \ge 0$; but $\alpha = 0$ corresponds to Hölder's inequality and thus is of no interest in this connection.

Set $\varphi = |f|$ and form

$$L(\varphi) = \varphi - \lambda x^\alpha \varphi^{\beta_1} - \mu x^\alpha \varphi^{\beta_2}, \tag{3}$$

where λ and μ are positive parameters at our disposal. Take the maximum of $L(\varphi)$ for fixed x and variable φ; let it be attained for $\varphi = \varphi_0(x)$. If we put $\int_a^b x^\alpha \varphi_0^{\beta_1} dx = A_1$ and $\int_a^b x^\alpha \varphi_0^{\beta_2} dx = A_2$, it is evident that among all functions φ giving the same values to these integrals the function $\varphi_0(x)$ gives the maximum of $\int_a^b \varphi\,dx$. The maximum of $L(\varphi)$ for fixed x is either $0 = L(0)$ or positive; in the latter case φ_0 is a solution of the equation

$$1 - \lambda\beta_1 x^\alpha \varphi_0^{\beta_1 - 1} - \mu\beta_2 x^\alpha \varphi_0^{\beta_2 - 1} = 0 \quad \text{or} \quad \lambda\beta_1\varphi_0^{\beta_1-1} + \mu\beta_2\varphi_0^{\beta_2-1} = x^{-\alpha}. \tag{4}$$