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On the uniform convexity of L^p and l^p

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CLARKSON defined in 1936 the uniformly convex spaces [2]. The uniform convexity asserts that there is a function $\delta(\varepsilon)$ of $\varepsilon > 0$ such that $\|x\| = 1$, $\|y\| = 1$, and $\|x - y\| \geqq \varepsilon$ imply $\|\frac{1}{2}(x+y)\| \leqq 1 - \delta(\varepsilon)$, where x and y are elements of the space. CLARKSON proved that the well-known spaces L^p and l^p are uniformly convex if $p > 1$. The purpose of this note is to give the best possible function $\delta(\varepsilon)$ for these spaces, i.e. to find for each $p > 1$ and $\varepsilon > 0$

$$\inf\left(1 - \left\|\frac{x+y}{2}\right\|\right)$$

under the conditions $\|x\| = 1$, $\|y\| = 1$, $\|x - y\| \geqq \varepsilon$. We need two inequalities, which are given in Theorem 1, formula (1). I have been informed that the left-hand side inequality of this formula was proved by BEURLING at a seminar in Uppsala in 1945, but it does not seem to be in print. The right-hand side inequality is proved by CLARKSON ([2] p. 400) and BOAS ([1] p. 305). We give here a reconstruction of BEURLING's proof and also for completeness a simple proof of the other inequality.

Let the functions in L^p be defined over $0 \leqq t \leqq 1$. The norm of $x = x(t)$ is then given by

$$\|x\|^p = \int_0^1 |x(t)|^p \, dt.$$

In l^p the norm of $x = (x_1, x_2, \ldots)$ is given by

$$\|x\|^p = \sum_{i=1}^{\infty} |x_i|^p.$$

Theorem 1. *For $p > 2$ the following inequalities hold*

$$(\|x\| + \|y\|)^p + |\|x\| - \|y\||^p \geqq \|x+y\|^p + \|x-y\|^p \geqq 2\|x\|^p + 2\|y\|^p. \tag{1}$$

For $1 < p < 2$ these inequalities hold in the reverse sense.

The equality sign holds for L^p [for l^p] in the left-hand side of (1) if and only if $x = 0$, or $y = 0$, or there is a number $a > 0$ such that $(x(t) - a\,y(t))\,(x(t) + a\,y(t)) = 0$ for almost every t [such that $(x_i - a\,y_i)\,(x_i + a\,y_i) = 0$ for every i], and in the right-hand side of (1) if and only if $x(t)\,y(t) = 0$ for almost every t [$x_i\,y_i = 0$ for every i].