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Meromorphic Szegő functions and asymptotic series for Verblunsky coefficients

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1. Introduction

This paper is concerned with the spectral theory of orthogonal polynomials on the unit circle (OPUC) [16], [17], [15], [23], [7], [8] in the case of particularly regular measures. Throughout, we will consider probability measures on $\partial\mathbf{D}=\{z\mid |z|=1\}$ of the form

$$d\mu=w(\theta)\,\frac{d\theta}{2\pi}+d\mu_{\mathrm{s}}, \tag{1.1}$$

where w obeys the Szegő condition, that is,

$$\int_0^{2\pi}\log(w(\theta))\,\frac{d\theta}{2\pi}>-\infty. \tag{1.2}$$

In that case, the Szegő function is defined by

$$D(z)=\exp\biggl(\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log(w(\theta))\,\frac{d\theta}{4\pi}\biggr). \tag{1.3}$$

Not only does w determine D, but D determines w, since $\lim_{r\uparrow 1}D(re^{i\theta})\equiv D(e^{i\theta})$ exists for a.e. θ and

$$w(\theta)=|D(e^{i\theta})|^2. \tag{1.4}$$

Indeed, D is the unique function analytic on $\mathbf{D}=\{z\mid |z|<1\}$ with $D(0)>0$ and D non-vanishing on $\mathbf{D}$ so that (1.4) holds.

Given $d\mu$, we let Φ_n be the monic orthogonal polynomial and $\varphi_n=\Phi_n/\|\Phi_n\|_{L^2(d\mu)}$. The Φ_n's obey the Szegő recursion

$$\Phi_{n+1}(z)=z\Phi_n(z)-\bar{\alpha}_n\Phi_n^*(z), \tag{1.5}$$

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