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A Carleman type theorem for proper holomorphic embeddings

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1. Introduction

We denote by $\mathbf{C}$ the field of complex numbers and by $\mathbf{R}$ the field of real numbers. To motivate our main result we recall the Carleman approximation theorem [4], [11]: *For each continuous function* $\lambda: \mathbf{R} \to \mathbf{C}$ *and positive continuous function* $\eta: \mathbf{R} \to (0, \infty)$ *there exists an entire function* f *on* $\mathbf{C}$ *such that* $|f(t) - \lambda(t)| < \eta(t)$ *for all* $t \in \mathbf{R}$. If λ is smooth, we can also approximate its derivatives by those of f. A more general result was proved by Arakelian [2] (see [14] for a simple proof).

Let $\mathbf{C}^n$ be the complex Euclidean space of dimension n. Our main result is an extension of Carleman's theorem to *proper holomorphic embeddings* of $\mathbf{C}$ into $\mathbf{C}^n$ for $n > 1$:

1.1. Theorem. *Let* $n > 1$ *and* $r \geq 0$ *be integers. Given a proper embedding* $\lambda: \mathbf{R} \hookrightarrow \mathbf{C}^n$ *of class* C^r *and a continuous positive function* $\eta: \mathbf{R} \to (0, \infty)$, *there exists a proper holomorphic embedding* $f: \mathbf{C} \hookrightarrow \mathbf{C}^n$ *such that*

$$|f^{(s)}(t) - \lambda^{(s)}(t)| < \eta(t), \quad t \in \mathbf{R},\ 0 \leq s \leq r.$$

If in addition $T = \{t_j\} \subset \mathbf{R}$ *is discrete, there exists* f *as above such that*

$$f^{(s)}(t) = \lambda^{(s)}(t), \quad t \in T,\ 0 \leq s \leq r.$$

Definition. Two proper holomorphic embeddings $f, g: \mathbf{C} \hookrightarrow \mathbf{C}^n$ are said to be $\operatorname{Aut} \mathbf{C}^n$-equivalent if $\Phi \circ f = g$ for some holomorphic automorphism Φ of $\mathbf{C}^n$.

1.2. Corollary. *For each* $n > 1$ *the set of* $\operatorname{Aut} \mathbf{C}^n$*-equivalence classes of proper holomorphic embeddings* $\mathbf{C} \hookrightarrow \mathbf{C}^n$ *is uncountable.*

For $n \geq 3$ the corollary is due to Rosay and Rudin [16]. The corollary follows from Theorem 1.1 and a result of Rosay and Rudin [15] to the effect that for each