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Relative vanishing theorems in characteristic p

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Introduction

In [I], L. Illusie proved a decomposition theorem of the relative de Rham complex for a morphism $f\colon X\to Y$ of smooth schemes defined over a perfect field k of characteristic $p>0$, generalizing an earlier result of [DI] which treats the absolute case. He deduced from the theorem several vanishing results for the direct image of line bundles.

Let $f\colon X\to Y$ be a k-morphism and E a vector bundle on X. In this paper we introduce the *p-cohomological dimension relative to* f, which will be denoted by $\operatorname{pcd}(E,f)$, of E. By means of this notion, we extend some of the vanishing theorems obtained in [I] to the case of higher rank bundles. As in [I], we need the assumption that f is semistable along a normal crossing divisor $D_Y\subset Y$ and f is liftable to $W_2(k)$, the ring of length two Witt vectors.

In Section 1, we prove a vanishing of direct image sheaves of vector bundles for a semistable morphism. The cohomology vanishing of the Gauss–Manin systems will be considered in Section 2. In Section 3, we treat the case of open varieties and generalize a theorem in [BK2] to the relative situation.

1. Relative vanishing for semistable morphisms

Let k be a perfect field of characteristic $p>0$. Let X be a smooth scheme defined over k. We denote by $F_X\colon X\to X$ the absolute Frobenius of X. A vector bundle E on X yields a bundle E' on X'. Let $F^nE:=(F_X^*)^nE$ denote the bundle on X obtained by the nth iterated pull-back of E by F_X.

Let Y be a smooth scheme over k and $f\colon X\to Y$ a k-morphism. We define the *p-cohomological dimension relative to* f of E, which we denote by $\operatorname{pcd}(E,f)$, as the smallest integer $\alpha\geq 0$ such that for every coherent sheaf $\mathcal{F}$, there exists $n_0=n_0(\mathcal{F})$ satisfying $R^kf_*(\mathcal{F}\otimes F^nE)=0$ for all $n\geq n_0$ and all $k>\alpha$.