ON A THEOREM OF DAVENPORT AND HEILBRONN

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1. Let $\lambda_1, \dots, \lambda_5$ be any 5 real numbers, not all of the same sign and none of them 0. It was proved by Davenport and Heilbronn [7] that, for any $\varepsilon > 0$, the inequality

$$|\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2| < \varepsilon \tag{1}$$

is soluble in integers $x_1, \dots, x_5$, not all 0. The mention of ε here is in reality superfluous, for if the solubility of

$$|\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2| < 1 \tag{2}$$

is proved for all $\lambda_1, \dots, \lambda_5$, then the solubility of (1) for any $\varepsilon > 0$ follows, on applying (2) with λ_j replaced by $\varepsilon^{-1}\lambda_j$.

Our object in the present paper is to make this result more precise by giving an estimate for a solution of (2) in terms of $\lambda_1, \dots, \lambda_5$. It would not be easy to do this with much precision by following the original line of argument, which depended on considering the continued fraction development of one of the ratios λ_i/λ_j.

The result we shall prove is as follows.

THEOREM. *For $\delta > 0$ there exists C_δ with the following property. For any real $\lambda_1, \dots, \lambda_5$, not all of the same sign and all of absolute value* 1 *at least, there exist integers $x_1, \dots, x_5$ which satisfy both* (2) *and*

$$0 < |\lambda_1| x_1^2 + \cdots + |\lambda_5| x_5^2 < C_\delta |\lambda_1 \dots \lambda_5|^{1+\delta}. \tag{3}$$

In another paper [1] we have applied this result to general indefinite quadratic forms. We have proved that any real indefinite quadratic form in 21 or more variables assumes values arbitrarily near to 0, provided that when the form is expressed