Commutators of Littlewood–Paley sums

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Introduction

For every interval $I \subset \mathbf{R}$ we denote by S_I the partial sum operator:

$$(S_I f)^\wedge = \hat{f} \mathcal{X}_I.$$

Given a sequence $\{I_j\}$ of disjoint intervals and a function b, we form the square function

$$[\Delta, b] f(x) = \left(\sum_j \left| b(x) S_{I_j} f(x) - S_{I_j}(bf)(x) \right|^2 \right)^{1/2}.$$

We aim to prove inequalities of the type

$$\left\| [\Delta, b] f \right\|_{L^p(\beta)} \leq C_p \| f \|_{L^p(\alpha)},$$

for some classes of weights α, β, depending on the family $\{I_j\}$ and on the function b. See Theorem (3.2) and (3.5).

Inequalities of the aforementioned type are new, even in the unweighted case, for general families of intervals $\{I_j\}$. In the case of the family of dyadic intervals some results are known see [ST2], for the smooth operators $\widetilde{S}_{I_j}$, see Definition (3.11).

We shall need a vector-valued commutator theorem (see Theorem (2.2)) for a kind of vector-valued L^r-Dini singular integrals. The use of these vector-valued L^r-Dini singular integrals in the Littlewood–Paley theory was introduced in the beautiful paper of J.L. Rubio de Francia [RF2].

To prove the commutator theorem, we shall need an extrapolation theorem (see Section 1) for pairs of weights α and β that satisfy the relation $\alpha = \nu^p \beta$, where ν is a given positive function and α and β belong to A_p. For notation and the general theory of A_p weights, we indicate [GCRF] for instance.