A simplified proof for a moving boundary problem for Hele-Shaw flows in the plane

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0. Introduction

In [7] Richardson derived a mathematical model for describing Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel. This model can be represented in the following form (see also [3]): Given $f_0(z)$, $f_0(0)=0$, analytic and univalent in a neighbourhood of $|z|\leq 1$, find $f(z,t)$, analytic and univalent as a function of z in a neighbourhood of $|z|\leq 1$, continuously differentiable with respect to t in a right-sided neighbourhood of $t=0$, satisfying

$$\text{(1)} \qquad \operatorname{Re}\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\overline{\frac{\partial f}{\partial z}(z,t)}\right)=1 \qquad \text{for } |z|=1;$$

$$\text{(2)} \qquad f(z,0)=f_0(z) \quad \text{for } |z|\leq 1;$$

$$\text{(3)} \qquad f(0,t)=0.$$

With the results of Vinogradov–Kufarev [9] one gets the existence and uniqueness of solutions which depend analytically on z and t under the additional assumption $f_z(0,t)>0$. But the proofs in [9] are fairly complicated.

For this reason Gustafsson gave in [3] a more elementary proof of existence and uniqueness of solutions of (1)–(3) in the case that $f_0(z)$ is a polynomial or a rational function. In both cases the solution is of the same sort with regard to z as the initial value $f_0(z)$. The restriction to rational initial values seems to be indispensable for the used reduction of (1) to a finite system of ordinary differential equations in t.

The goal of the present paper is to give a simplified proof for a generalized Hele-Shaw problem containing as a special case the above formulated problem (1)–(3). This proof is based on the application of the non-linear abstract Cauchy–Kovalevsky theorem which was proved by Nishida in [5]. Moreover, this theorem gives uniqueness for solutions depending continuously differentiably on t.