Wavelets and paracommutators

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1. Introduction

Let P denote the collection of all dyadic cubes in $\mathbf{R}^d$, $E=\{0,1\}^d\setminus\{0,0\}$ and $\Lambda=P\times E$. The construction of the wavelet bases $\{\psi_\lambda\}_{\lambda\in\Lambda}$ on $L^2(\mathbf{R}^d)$ is a celebrated result in mathematical analysis. Y. Meyer [7] and G. David [3] have used the bases $\{\psi_\lambda\}$ to study the boundedness of the Calderón–Zygmund operators, and have successfully simplified the proofs of the "$T1$ Theorem" and the "Tb Theorem". It is well known that operators of the form

$$Tf(x)=\int_{R^d} K(x,y)f(y)\,dy$$

can be compact; for example $T\in S_2$, i.e. T is Hilbert–Schmidt, if and only if $K\in L^2(\mathbf{R}^d\times\mathbf{R}^d)$. In this paper we will study the compactness and the Schatten–von Neumann properties by wavelet bases.

Precisely speaking, we consider the bilinear form $T(f,g)=\langle T(f),g\rangle$ on $\mathcal{D}\times\mathcal{D}$ with the distributional kernel $K(x,y)$. Let $\alpha_{\lambda,\lambda'}=T(\psi_\lambda,\psi_{\lambda'})$, for $\lambda,\lambda'\in\Lambda$. Then the bilinear form T is determined by the infinite matrix $\{\alpha_{\lambda,\lambda'}\}$. In other words, $\sum_\lambda\sum_{\lambda'}\alpha_{\lambda,\lambda'}\psi_\lambda\otimes\psi_{\lambda'}$ gives a decomposition for T, called the standard decomposition. We will also consider non-standard decompositions for T. And we will find out the conditions for $T\in S_p$ (the Schatten–von Neumann class). Then we will consider an important example: paracommutators , which are defined and studied systematically by Janson and Peetre [5]. Supplementary results are given by Peng [9], [10], and Peng–Qian [11].

In Section 2 we will give some notations and definitions, and in Section 3 we will give some lemmas for the S_p estimates. In Section 4 we will revisit the paracommutator, and will give simplified proofs for most results on paracommutators by wavelet basis expansions.

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