Riemann's zeta-function and the divisor problem. II

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1. Introduction

An interesting analogy between the divisor function $d(n)$ and the function $|\zeta(\frac{1}{2}+it)|^2$ was pointed out by F.V. Atkinson [1] fifty years ago. A celebrated result in this direction is Atkinson's formula [2] for the error term $E(T)$ in the relation

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = (\log(T/2\pi)+2\gamma-1)T+E(T),$$

where γ denotes Euler's constant. The most significant terms in this formula are—up to an oscillating sign—similar to those in Voronoi's formula for the error term $\Delta(x)$ in Dirichlet's divisor problem for the sum

$$\sum_{n\le x} d(n) = (\log x+2\gamma-1)x+\Delta(x).$$

More precisely, $E(T)$ is comparable with $2\pi\Delta(T/2\pi)$ in this sense. In [7] we showed that $E(T)$ should actually be compared with $2\pi\Delta^*(T/2\pi)$, where

$$\Delta^*(x) = -\Delta(x)+2\Delta(2x)-\tfrac{1}{2}\Delta(4x),$$

because the Voronoi formula for $2\pi\Delta^*(T/2\pi)$ is analogous to Atkinson's formula for $E(T)$ even as to the signs of the terms.

The function $\Delta^*(x)$ can be understood as the error in a certain divisor problem. Namely, it was observed by T. Meurman [10] that

$$\tfrac{1}{2}\sum_{n\le 4x} (-1)^n d(n) = (\log x+2\gamma-1)x+\Delta^*(x).$$