Acta Math., 176 (1996), 49–71

The xi function

by

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1. Introduction

Despite the fact that spectral and inverse spectral properties of one-dimensional Schrödinger operators $H=-d^2/dx^2+V$ have been extensively studied for seventy-five years, there remain large areas where our knowledge is limited. For example, while the inverse theory for operators on $L^2((-\infty,\infty))$ is well understood in case V is periodic [12], [24], [25], [35], [39]–[42], [49], it is not understood in case $\lim_{|x|\to\infty} V(x)=\infty$ and H has discrete spectrum.

Our goal here is to introduce a special function $\xi(x,\lambda)$ on $\mathbf{R}\times\mathbf{R}$ associated to H which we believe will be a valuable tool in the spectral and inverse spectral theory. In a sense we will make precise, it complements the Weyl m-functions, $m_\pm(x,\lambda)$.

A main application of ξ which we will make here concerns a generalization of the trace formula for Schrödinger operators to general V's.

Recall the well-known trace formula for periodic potentials: Let $V(x)=V(x+1)$. Then, by Floquet theory (see, e.g., [10], [37], [44]),

$$\operatorname{spec}(H)=[E_0,E_1]\cup[E_2,E_3]\cup\ldots,$$

a set of bands. If V is C^1, one can show that the sum of the gap sizes is finite, that is,

$$\sum_{n=1}^{\infty}|E_{2n}-E_{2n-1}|<\infty. \tag{1.1}$$

For a fixed y, let H_y be the operator $-d^2/dx^2+V$ on $L^2([y,y+1])$ with $u(y)=u(y+1)=0$ boundary conditions. Its spectrum is discrete, that is, there are eigenvalues $\{\mu_n(y)\}_{n=1}^{\infty}$ with

$$E_{2n-1}\leqslant\mu_n(y)\leqslant E_{2n}. \tag{1.2}$$

This material is based upon work supported by the National Science Foundation under Grant DMS-9101715. The Government has certain rights in this material.