MAXIMAL ALGEBRAS OF CONTINUOUS FUNCTIONS

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1. Introduction

Let X be a compact Hausdorff space and $C(X)$ the algebra of all continuous complex-valued functions on X. Let A be a uniformly closed complex linear subalgebra of $C(X)$. Our interest centers about such algebras A which are maximal among all proper closed subalgebras of $C(X)$. In this paper we gather together most of the known facts concerning maximal algebras, give some new results, and some new proofs of known theorems.

A major motivation for the study of maximal algebras stems from an attempt to generalize the Stone-Weierstrass approximation theorem to non-self-adjoint algebras. This theorem states that if A is a self-adjoint closed subalgebra of $C(X)$ ($f \in A$ implies $\bar{f} \in A$), and if A separates points and contains the constant function 1, then $A = C(X)$. See [11; p. 8] for a proof. This result can be restated as follows: (i) every proper self-adjoint closed algebra A is contained in a self-adjoint maximal algebra and (ii) the self-adjoint maximal algebras, B, are of two kinds; either $B = [f \in C(X), f(x_0) = 0]$ for a fixed $x_0 \in X$, or $B = [f \in C(X); f(x_1) = f(x_2)]$ for fixed $x_1, x_2 \in X$. The condition, A contains the function 1, says that A is not in a maximal algebra of the first kind. The condition, A separates points, says that A is not in a maximal algebra of the second kind. Thus A is not contained in any self-adjoint maximal algebra and consequently, from (i), A is not a proper subalgebra, i.e., $A = C(X)$. A refinement of the Stone-Weierstrass theorem classifies all self-adjoint closed subalgebras of $C(X)$ and says that such an algebra A is the algebra of all continuous functions on an identification space of X, with the common zeros of the functions in A deleted. This result can be reinterpreted as saying that A is the intersection of the self-adjoint maximal algebras which contain it.

Let us drop the self-adjointness condition on A. One might now hope that the way to generalize the Stone-Weierstrass theorem would be to show that (i) holds (with self-

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