SIMILARITY OF OPERATOR ALGEBRAS

BY

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Dedicated to M. H. Stone on his seventy fifth birthday

1. Introduction

When viewed in a certain light, Tomita's theorem (the main result of the Tomita–Takesaki theory—see [3, 14, 15, 16, 17]) appears as the combination of a result on "unbounded" similarity between self-adjoint operator algebras and the special structure of a von Neumann algebra and its commutant relative to a joint separating vector. The main purpose of this article is to introduce and develop the theory of such similarities. (See section 3.) Our secondary purpose is to present a full proof of Tomita's theorem in the style mentioned. (See section 4.) In connection with this argument, we develop a new density result (Theorem 4.10). In section 2 we prove a bounded similarity result.

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2. Bounded similarity

If $\mathcal{H}$ is a complex Hilbert space and H is an operator on $\mathcal{H}$ such that $0 < aI \leqslant H \leqslant bI$, then H is bounded and sp (H), the spectrum of H, lies in $[a, b]$. In addition, H has an inverse with spectrum in $[b^{-1}, a^{-1}]$. If $\varphi(T) = HTH^{-1}$ for T in $\mathcal{B}(\mathcal{H})$, then φ is a bounded operator on $\mathcal{B}(\mathcal{H})$ and sp (φ) (relative to $\mathcal{B}(\mathcal{B}(\mathcal{H}))$) is contained in $[ab^{-1}, a^{-1}b]$. To see this, note that left multiplication by H on $\mathcal{B}(\mathcal{H})$ has the same spectrum as H, that right multiplication by H^{-1} has the same spectrum as H^{-1}, and that these two multiplications commute.

We employ the Banach-algebra-valued, holomorphic function calculus (see, for example, [1; Chapter VII]) to discuss holomorphic functions f of an element A of a Banach