SOME PROBLEMS OF DIOPHANTINE APPROXIMATION.

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I.

The fractional part of $n^k\theta$.

1.0 — Introduction.

1.00. Let us denote by $[x]$ and (x) the integral and fractional parts of x, so that

$$(x) = x - [x], \qquad 0 \leq (x) < 1.$$

Let θ be an irrational number, and α any number such that $0 \leq \alpha < 1$. Then it is well known that it is possible to find a sequence of positive integers $n_1, n_2, n_3, \cdots$ such that

$$(n_r\theta) \to \alpha \tag{1.001}$$

as $r \to \infty$.

It is necessary to insert a few words of explanation as to the meaning to be attributed to relations such as (1.001), here and elsewhere in the paper, in the particular case in which $\alpha = 0$. The formula (1.001), when $\alpha > 0$, asserts that, given any positive number ε, we can find r_0 so that

$$-\varepsilon < (n_r\theta) - \alpha < \varepsilon \qquad (r > r_0).$$

The points $(n_r\theta)$ may lie *on either side of* α. But $(n_r\theta)$ is never negative, and so, in the particular case in which $\alpha = 0$, the formula, if interpreted in the obvious manner, asserts *more* than this, viz. that

$$0 \leq (n_r\theta) < \varepsilon \qquad (r > r_0).$$