Acta Math., 175 (1995), 197–226

Everywhere discontinuous harmonic maps into spheres

by

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1. Introduction

Let Ω be a bounded domain of $\mathbf{R}^n$, S^2 the unit sphere of $\mathbf{R}^3$ and (Σ, h) a surface homeomorphic to S^2 with a metric h. We may assume, using the Nash–Moser theorem, that Σ is isometrically imbedded in some Euclidean space $\mathbf{R}^k$. We consider the Sobolev space

$$H^1(\Omega,\Sigma)=\{u\in H^1(\Omega,\mathbf{R}^k): u(x)\in\Sigma, \text{ a.e. } x\in\Omega\}.$$

Let $E(u)=\int_\Omega|\nabla u|^2$ be the Dirichlet energy for any u in $H^1(\Omega,\Sigma)$. For a sufficiently small neighborhood V of Σ in $\mathbf{R}^k$ the projection π of a point x of V is well defined. Weakly harmonic maps from Ω into Σ are critical points in $H^1(\Omega,\Sigma)$ of the Dirichlet energy in the following way:

$$u \text{ is weakly harmonic if } \forall\xi\in C_c^\infty(\Omega,\mathbf{R}^n),\ \frac{d}{dt}E(\pi(u+t\xi))\Big|_{t=0}=0. \tag{1}$$

This is equivalent to the fact that u verifies the Euler–Lagrange equation

$$-\Delta u=A(u)(\nabla u,\nabla u) \text{ in } \mathcal{D}'(\Omega,\mathbf{R}^m),\quad u\in\Sigma \text{ a.e.}, \tag{2}$$

where $A(u)$ is the second fundamental form of Σ and where we have used the notation

$$A(u)(\nabla u,\nabla u)=A(u)\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial x}\right)+A(u)\left(\frac{\partial u}{\partial y},\frac{\partial u}{\partial y}\right)+A(u)\left(\frac{\partial u}{\partial z},\frac{\partial u}{\partial z}\right).$$

In this paper we are interested in the singular set of such maps.

The system (2) is non-linear elliptic, and the non-linearity, produced by the fact that our map takes its values in a non-flat manifold, has a quadratic growth for the gradient:

$$-\Delta u=f(u,\nabla u) \text{ in } \mathcal{D}'(\Omega,\mathbf{R}^m) \quad\text{where } |f(x,p)|\leqslant C(|x|^s+|p|^2). \tag{3}$$