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Theta functions and Siegel–Jacobi forms

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1. Introduction

The main theme of this paper is the computation of rings of Siegel modular forms (of arbitrary level) and rings of Jacobi forms using an algebraic method. In the case of Siegel modular forms we get a generalization of results of Igusa. Our main tool is to introduce other theta functions, which are easier to handle and are more general than the classical one. As a geometric application we give a description of the Shioda surfaces and a compactification (even a projective variety) of the universal abelian variety. This leads to a result about Jacobi forms similar to Igusa's fundamental lemma for modular forms.
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2. Siegel modular forms of higher level

Throughout the paper we will use the same notation as in [R1], [R2]. For general facts we refer to [Ig3], [Kr], [Wi]. So let

$$\begin{aligned}\mathbf{H}_g &= \{\tau \in \mathrm{Mat}_{g\times g}(\mathbf{C}) \mid \tau \text{ symmetric}, \mathrm{Im}(\tau)>0\},\\ \Gamma_g &= \mathrm{Sp}(2g,\mathbf{Z}),\\ \Gamma_g(n) &= \mathrm{Ker}(\Gamma_g \to \mathrm{Sp}(2g,\mathbf{Z}/n)).\end{aligned}$$

For a subgroup of finite index $\Gamma\subset\Gamma_g$, we denote by $A(\Gamma)=\bigoplus_k[\Gamma,k]$ the ring of modular forms for Γ and by $\mathcal{A}_g(\Gamma)=\mathrm{Proj}(A(\Gamma))$ the corresponding Satake compactification, which contains $\mathbf{H}_g/\Gamma$ as an open dense subset. The open part $\mathbf{H}_g/\Gamma$ is the coarse moduli space for principally polarized abelian varieties with level-Γ structure.

We recall the classical notation for theta functions, i.e.

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau,z)=\sum_{x\in\mathbf{Z}^g}\exp 2\pi i\left(\tfrac12\tau\left[x+\tfrac12\alpha\right]+\left\langle x+\tfrac12\alpha, z+\tfrac12\beta\right\rangle\right)$$