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Painlevé's problem and the semiadditivity of analytic capacity

by

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1. Introduction

The *analytic capacity* of a compact set $E \subset \mathbf{C}$ is defined as

$$\gamma(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all analytic functions $f: \mathbf{C} \setminus E \to \mathbf{C}$ with $|f| \leqslant 1$ on $\mathbf{C} \setminus E$, and $f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty))$. For a general set $F \subset \mathbf{C}$, we set $\gamma(F) = \sup\{\gamma(E): E \subset F, E \text{ compact}\}$.

The notion of analytic capacity was first introduced by Ahlfors [Ah] in the 1940's in order to study the removability of singularities of bounded analytic functions. A compact set $E \subset \mathbf{C}$ is said to be removable (for bounded analytic functions) if for any open set Ω containing E, every bounded function analytic on $\Omega \setminus E$ has an analytic extension to Ω. In [Ah] Ahlfors showed that E is removable if and only if $\gamma(E) = 0$. However, this result doesn't characterize removable singularities for bounded analytic functions in a geometric way, since the definition of γ is purely analytic.

Analytic capacity was rediscovered by Vitushkin in the 1950's in connection with problems of uniform approximation of analytic functions by rational functions (see [Vi], for example). He showed that analytic capacity plays a central role in this type of problems. This fact motivated a renewed interest in analytic capacity. The main drawback of Vitushkin's techniques arises from the fact that there is not a complete description of analytic capacity in metric or geometric terms.

On the other hand, the *analytic capacity* γ_+ (or *capacity* γ_+) of a compact set E is

$$\gamma_+(E) = \sup_{\mu} \mu(E),$$

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