Counterexamples to a conjecture of Grothendieck

by

GILLES PISIER

Université Paris VI, Paris, France

In his thesis ([7] II. p. 136) and in his fundamental paper ([6] p. 74), Grothendieck formulated the following conjecture: If two Banach spaces X and Y are such that their injective and projective tensor products $X\check{\otimes}Y$ and $X\hat{\otimes}Y$ coïncide, then either X or Y must be finite dimensional. The aim of this paper is to give a counterexample.

We will exhibit a separable infinite dimensional Banach space X such that $X\check{\otimes}X=X\hat{\otimes}X$, both algebraically and topologically. The space X is of cotype 2 as well as its dual. Moreover, the natural map from $X^*\hat{\otimes}X$ into $X^*\check{\otimes}X$ is surjective, but it is not injective, since X fails the approximation property (in short the A.P.); equivalently, every operator on X which is a uniform limit of finite rank operators is nuclear. This implies that there are (roughly) "very few" operators on X of finite rank and of small norm. For instance, there is a number $\delta>0$ such that, for any finite dimensional subspace E of X and for any projection $P: X\to E$, we have

$$\|P\| \geq \delta(\dim E)^{1/2}.$$

Therefore, if $\{P_n\}$ is a sequence of finite rank projections on X, then $\|P_n\|$ must tend to infinity if the rank of P_n tends to infinity. A fortiori, the space X can contain uniformly complemented l_p^n's for *no p* such that $1\leq p\leq\infty$, so that we have also a negative answer to a question of Lindenstrauss [13].

Finally, since X is not isomorphic to a Hilbert space, although X and X^* are both of cotype 2 we also answer negatively a question raised by Maurey in [17] (as well as question 5.3 in [4]). Moreover, our example shows that the A.P. cannot be removed from the assumptions of the factorization theorem of [23].

In the last ten years, under the impulse of [14], several significant steps were taken towards the solution of Grothendieck's conjecture; besides [22] and [23], the results of the papers [17], [10] and [1] play an important rôle (directly or indirectly) in our construction. During the same period, Grothendieck's conjecture was established