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A partial description of parameter space of rational maps of degree two: Part I

by

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1. Introduction

Rational maps have been much studied as dynamical systems. Many rational maps are hyperbolic, and easy to analyse. More importantly, variation of dynamics even within a one-parameter family of rational maps is usually extremely rich, and can sometimes be described in great detail. The prime example is the family of quadratic polynomials $\{z^2+a: a\in\mathbf{C}\}$ which has been the subject of a fundamental study by Douady and Hubbard [D], [D–H1], [D–H2], some of which has been reinterpreted by Thurston [T]. This is, in fact, the main motivation for the present work, which is concerned with the family of rational maps of degree two. The aim is to understand the variation of dynamics within this family.

It has been clear since the pioneering work of Fatou and Julia [F], [J] that the dynamics of a rational map is largely influenced by—and sometimes completely determined by—the dynamics of its critical points. This, in itself, is an example of a vague, but recurrent, theme in dynamical systems in general—that the variation of dynamics in some family of maps (or flows) should be determined by the movement of some (hopefully finite) set of points—which might be periodic, or homoclinic, or, as in the present example, critical. Thus, in the family of polynomials $\{z^2+a: a\in\mathbf{C}\}$, it is the behaviour of the critical point 0 which is important, and in the family of rational maps of degree two, it is the behaviour of the two critical points. The interesting dynamics of a rational map occur on its Julia set. Variation of dynamics in families of rational maps is visible even in static pictures, since the structure of the Julia set often changes radically, even up to homeomorphism, under changes in parameter. However (as always with dynamical systems), dynamics, and structure of the Julia set, are constant on hyperbolic components. The union of hyperbolic components is conjectured to be dense in the family of rational or polynomial maps of degree d, for any d.