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On the complexity of the classification problem for torsion-free abelian groups of rank two

by

SIMON THOMAS

Rutgers University
Piscataway, NJ, U.S.A.

1. Introduction

This paper is a contribution to the project [9], [8], [1], [13] of explaining why no satisfactory system of complete invariants has yet been found for the torsion-free abelian groups of finite rank $n \geqslant 2$. Recall that, up to isomorphism, the torsion-free abelian groups of rank n are exactly the additive subgroups of the n-dimensional vector space $\mathbf{Q}^n$ which contain n linearly independent elements. Thus the collection of torsion-free abelian groups of rank $1 \leqslant r \leqslant n$ can be naturally identified with the set $S(\mathbf{Q}^n)$ of all non-trivial additive subgroups of $\mathbf{Q}^n$. In 1937, Baer [3] solved the classification problem for the class $S(\mathbf{Q})$ of rank-one groups as follows.

Let $\mathbf{P}$ be the set of primes. If G is a torsion-free abelian group and $0 \neq x \in G$, then the *p-height* of x is defined to be

$$h_x(p) = \sup\{n \in \mathbf{N} \mid \text{there exists } y \in G \text{ such that } p^n y = x\} \in \mathbf{N} \cup \{\infty\};$$

and the *characteristic* $\chi(x)$ of x is defined to be the function

$$\langle h_x(p) \mid p \in \mathbf{P} \rangle \in (\mathbf{N} \cup \{\infty\})^{\mathbf{P}}.$$

Two functions $\chi_1, \chi_2 \in (\mathbf{N} \cup \{\infty\})^{\mathbf{P}}$ are said to be *similar* or to *belong to the same type*, written $\chi_1 \equiv \chi_2$, if and only if

(a) $\chi_1(p) = \chi_2(p)$ for almost all primes p; and

(b) if $\chi_1(p) \neq \chi_2(p)$, then both $\chi_1(p)$ and $\chi_2(p)$ are finite.

Clearly $\equiv$ is an equivalence relation on $(\mathbf{N} \cup \{\infty\})^{\mathbf{P}}$. If G is a torsion-free abelian group and $0 \neq x \in G$, then the *type* $\tau(x)$ of x is defined to be the $\equiv$-equivalence class containing the characteristic $\chi(x)$.

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