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Boundary behavior of extremal plurisubharmonic functions

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Introduction

In a celebrated paper Lempert [12] and later in a more general setting Klimek [9], Poletskiĭ [20], and Zaharjuta [27] introduced the notion of a pluricomplex Green function g for a bounded convex domain G in $\mathbf{C}^N$. Assuming that G contains the origin, this Green function with pole at 0 is given by

$$g(z) := \sup_u u(z), \quad z \in G,$$

where the supremum is taken over all plurisubharmonic functions $u: G \to [-\infty, 0[$ with $u(w) \leqslant \log|w| + O(1)$ as $w \to 0$. This function is plurisubharmonic and it is continuous on $\overline{G} \setminus \{0\}$ if $g \,|\, \partial G :\equiv 0$. Lempert's results imply that also the sublevel sets $G_x = \{z \mid g(z) < x\}$, $x < 0$, are convex. If

$$H_x(z) := \sup\{\operatorname{Re}\langle w, z\rangle \mid g(w) < x\}, \quad z \in \mathbf{C}^N,$$

denotes the supporting function of $G_x \subset \mathbf{R}^{2N}$, we introduce a type of directional Lelong number

$$D_G(a) := \lim_{x \uparrow 0} \frac{H_0(a) - H_x(a)}{-x} \in\,]0, \infty], \quad a \in S := \{z \in \mathbf{C}^N \mid |z| = 1\},$$

which measures the rate of approximation of ∂G by ∂G_x, $x<0$, in the direction of a. In the case that there is a biholomorphic mapping ψ of the ball $U := \{z \in \mathbf{C}^N \mid |z| < 1\}$ onto G with $\psi(0) = 0$, this quantity is closely related to the notion of an angular derivative. For instance if $N = 1$, D_G is bounded if and only if $|\psi'|$ is bounded on G (see [16]).

We show that the lower semicontinuous function D_G is connected with the boundary behavior of another extremal plurisubharmonic function, which has been introduced by Siciak [23], [24], [25]. We put $H := H_0$ and consider

$$V_H(z) = \sup_u u(z), \quad z \in \mathbf{C}^N,$$