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# Integers, without large prime factors, in arithmetic progressions, I

by

ANDREW GRANVILLE([1])

*Institute for Advanced Study*
*Princeton, NJ, U.S.A.*

## 1. Introduction

Let $\Psi(x,y)$ be the number of integers $\leqslant x$ that are free of prime factors $>y$, $\Psi_q(x,y)$ be the number of such integers that are also coprime to $q$, and $\Psi(x,y;a,q)$ be the number of such integers in the congruence class $a \pmod q$. Good estimates for these functions have many applications in number theory (for instance, to bounds for the least quadratic non-residue $\pmod p$ [Bg], to Waring's problem [Va], to finding large gaps between primes [Ra] and to analysis of factoring algorithms [Po]), as well as being interesting in of themselves, and so have been extensively investigated.

Recently Hildebrand and Tenenbaum [HT] have provided a good estimate for $\Psi(x,y)$ for all $x \geqslant y \geqslant 2$, and a similar method works for $\Psi_q(x,y)$; however their method applies to $\Psi(x,y;a,q)$ only when $q$ is considerably smaller than $y$. In general one expects that, for sufficiently large $x$,

$$\Psi(x,y;a,q) \sim \frac{\Psi_q(x,y)}{\phi(q)} \tag{1.1}$$

whenever $a$ is coprime to $q$, provided that the primes $\leqslant y$ generate the full multiplicative group of units $\pmod q$. Buchstab [Bu] proved such a result when $q$ and $u$ $(:=\log x/\log y)$ are fixed; and extensions of this, for $q$ up to a fixed power of $\log y$, are considered in [LF] and [No]. Recently Fouvry and Tenenbaum [FT] have shown that the estimate

$$\Psi(x,y;a,q) = \frac{\Psi_q(x,y)}{\phi(q)}\bigl\{1+O\bigl(\exp\bigl(-c\sqrt{\log y}\,\bigr)\bigr)\bigr\} \tag{1.2}$$

holds uniformly for

$$y \geqslant 2, \quad x \geqslant y \geqslant \exp(c(\log\log x)^2), \tag{1.3}$$

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