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A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry

by

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Introduction

It is the purpose of this paper to introduce and study a nonlinear elliptic system of equations imposed on a map from a Hermitian into a Riemannian manifold which seems to be more appropriate to Hermitian geometry than the harmonic map system. Thus, let X be a complex manifold with Hermitian metric $(\gamma_{\alpha\bar\beta})$ in local coordinates, N a Riemannian manifold with metric (g_{ij}) and Christoffel symbols Γ^i_{jk}. A harmonic map $f\colon X\to N$ then has to satisfy

$$\frac{1}{2}\frac{\partial}{\partial z^{\bar\beta}}\left(\gamma^{\alpha\bar\beta}\frac{\partial f^i}{\partial z^\alpha}\right)+\frac{1}{2}\frac{\partial}{\partial z^\alpha}\left(\gamma^{\alpha\bar\beta}\frac{\partial f^i}{\partial z^{\bar\beta}}\right)+\gamma^{\alpha\bar\beta}\Gamma^i_{jk}(f(z))\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\bar\beta}}=0,\quad i=1,\dots,\dim N \tag{H1}$$

in local coordinates. A disadvantage of this system is that, unless X is Kähler, a holomorphic map need not be harmonic. We therefore replace (H1) by

$$\gamma^{\alpha\bar\beta}\left(\frac{\partial^2 f^i}{\partial z^\alpha\partial z^{\bar\beta}}+\Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial z^{\bar\beta}}\right)=0,\quad i=1,\dots,\dim N. \tag{H2}$$

We point out that (H1) and (H2) are equivalent if X is Kählerian. In general, (H2) is analytically more difficult than (H1) because it neither has a divergence nor a variational structure.

A vague analogue of the difference between (H1) and (H2) is given by the two different possibilities of defining geodesics on a manifold when the connection is not the Levi-Civita connection, i.e., not compatible with the metric. One can define geodesics metrically, namely as critical points for a length or energy integral, or via the connection,

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