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Multiplicities of algebraic linear recurrences

by

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1. Introduction

Let n be a natural number. We shall study linear recurrence sequences

$$u_{m+n}=\nu_{n-1}u_{m+n-1}+\nu_{n-2}u_{m+n-2}+\ldots+\nu_0u_m,\quad m=0,1,2,\ldots. \tag{1.1}$$

Here we assume that $\nu_{n-1},\ldots,\nu_0$ are elements of $\mathbf{C}$ with $\nu_0\neq0$. We assume moreover that the initial values $u_0,\ldots,u_{n-1}$ of our sequence have $|u_{n-1}|+\ldots+|u_0|>0$. Let

$$G(z)=z^n-\nu_{n-1}z^{n-1}-\ldots-\nu_0 \tag{1.2}$$

be the companion polynomial of the recurrence (1.1) and write

$$G(z)=\prod_{i=1}^{r}(z-\alpha_i)^{\varrho_i} \tag{1.3}$$

with distinct numbers $\alpha_1,\ldots,\alpha_r$. We call n the order and r the rank of the recurrence (1.1). Before we state our results, we shall recall a few facts about linear recurrence sequences. An excellent account on this topic may be found in the introductory Chapter C of Shorey and Tijdeman [13]. In the sequel we quote some of the theorems collected there.

Let $(u_m)_{m=0}^{\infty}$ be a sequence satisfying relation (1.1) *with $\nu_0\neq0$. For $i=1,\ldots,r$ let α_i and ϱ_i be determined by* (1.2) *and* (1.3) *where the numbers $\alpha_1,\ldots,\alpha_r$ are distinct. Then there exist uniquely determined polynomials $f_i\in\mathbf{Q}(u_0,\ldots,u_{n-1},\nu_0,\ldots,\nu_{n-1},\alpha_1,\ldots,\alpha_r)[z]$ of degree $\leqslant\varrho_i-1$ $(i=1,\ldots,r)$ such that*

$$u_m=\sum_{i=1}^{r}f_i(m)\alpha_i^m,\quad m=0,1,2,\ldots. \tag{1.4}$$