

# Quasiconformal maps in metric spaces with controlled geometry

by

JUHA HEINONEN

and

PEKKA KOSKELA

*University of Michigan  
Ann Arbor, MI, U.S.A.*

*University of Jyväskylä  
Jyväskylä, Finland*

## Contents

1. Introduction
2. Modulus and capacity in a metric space
3. Loewner spaces
4. Quasiconformality vs. quasisymmetry
5. Poincaré inequalities and the Loewner condition
6. Examples of Loewner spaces
7. Absolute continuity of quasisymmetric maps
8. Quasisymmetric invariance of Loewner spaces
9. Quasiconformal maps and Sobolev spaces

## 1. Introduction

This paper develops the foundations of the theory of quasiconformal maps in metric spaces that satisfy certain bounds on their mass and geometry. The principal message is that such a theory is both relevant and viable.

The first main issue is the problem of definition, which we next describe. Quasiconformal maps are commonly understood as homeomorphisms that distort the shape of infinitesimal balls by a uniformly bounded amount. This requirement makes sense in every metric space. Given a homeomorphism  $f$  from a metric space  $X$  to a metric space  $Y$ , then for  $x \in X$  and  $r > 0$  set

$$H_f(x, r) = \frac{\sup\{|f(x) - f(y)| : |x - y| \leq r\}}{\inf\{|f(x) - f(y)| : |x - y| \geq r\}}. \quad (1.1)$$

Here and hereafter we use the distance notation  $|x - y|$  in any metric space.

---

Both authors were supported in part by the NSF and the Academy of Finland. The first author is a Sloan Fellow.