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Classical area minimizing surfaces with real-analytic boundaries

by

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Let C be a smooth embedded closed curve in $\mathbf{R}^n$ (or more generally in an n-manifold with a real-analytic Riemannian metric) and let S be an area minimizing disk with boundary C. Then S can be parametrized by an almost-conformal map F from the closed unit disk $\mathbf{D}$ to $\mathbf{R}^n$. Almost-conformality of F means that F is conformal except for finitely many points at which DF vanishes. Such exceptional points are called branch points. Even though DF vanishes at a branch point p, there may be a neighborhood U of p such that $F(U)$ is a smooth embedded 2-manifold; that is, the image surface may be smooth even though the parametrization is not. If so, the branch point is called a false branch point. Otherwise it is called a true branch point. In [G2], R. Gulliver proved that F cannot have any false branch points. In this paper, we show that F cannot have true branch points along any real-analytic portion of the portion of the boundary curve C. This is somewhat surprising for the following reason. There are many examples of area minimizing disks in $\mathbf{R}^n$ (if $n\geqslant 4$) with interior true branch points, such as

$$z\in\mathbf{D}\subset\mathbf{C}\mapsto(z^3,z^{3k+1})\in\mathbf{C}^2\cong\mathbf{R}^4,$$

which is area minimizing by the Wirtinger inequality [F]. If S is such a surface and C' is a closed curve in S that passes through one of the branch points, then the portion of S bounded by C' will be an area minimizing disk with a true boundary branch point. In this way one can make, for any $k<\infty$, a $\mathcal{C}^k$-curve in $\mathbf{R}^4$ that bounds an area minimizing disk with a true boundary branch point. Moreover, R. Gulliver has pointed out that the example in [G3] is the real part of a holomorphic curve S in $\mathbf{C}^3\cong\mathbf{R}^6$; this surface S is area minimizing and has a $\mathcal{C}^\infty$-boundary curve with a true boundary branch point. However, according to our theorem, no such boundary curve can be real-analytic.

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