On the Thue-Siegel-Dyson theorem

by

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I. Introduction

I.1. The well-known Thue-Siegel theorem, in the refined form obtained by Dyson and Gelfond, asserts that if α is an algebraic number of degree $r \geqslant 2$ and if $\varepsilon > 0$ then

$$\left| \alpha - \frac{p}{q} \right| > q^{-\sqrt{2r} - \varepsilon}$$

for $q \geqslant q_0(\alpha, \varepsilon)$. The constant $q_0(\alpha, \varepsilon)$ in this result turns out to be not effectively computable.

In fact, Thue proved a result of this type with the exponent $r/2+1$, Siegel improved this to the exponent $\min(r/(s+1)+s)$ for $s=0, 1, \dots, r-1$ and finally, by using full freedom in the construction of the auxiliary polynomial, Dyson and Gelfond independently arrived at the exponent $\sqrt{2r}$.

The common feature in the approach of Thue, Siegel, Dyson and Gelfond is the consideration of two approximations $p_1/q_1, p_2/q_2$ to α and the construction of an auxiliary polynomial $p(x_1, x_2)$ with integral coefficients vanishing to a high order at (α, α) and vanishing only to a low order at $(p_1/q_1, p_2/q_2)$. Although Siegel and Schneider soon realized that further improvements could be obtained by the consideration of several distinct approximations $p_1/q_1, \dots, p_m/q_m$ to α and by the construction of an auxiliary polynomial $P(x_1, \dots, x_m)$ in many variables, it took about thirty years before Roth showed how to prove that P would vanish only to a low order at the point $(p_1/q_1, \dots, p_m/q_m)$. In this way Roth was able to prove his celebrated theorem

$$\left| \alpha - \frac{p}{q} \right| > q^{-2-\varepsilon}$$

for $q \geqslant q_0(\alpha, \varepsilon)$.