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Witt vectors of non-commutative rings and topological cyclic homology

by

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Introduction

Classically, one has for every commutative ring A the associated ring of p-typical Witt vectors $W(A)$. In this paper we extend the classical construction to a functor which associates to any associative ring A an abelian group $W(A)$. The extended functor comes equipped with additive Frobenius and Verschiebung operators. We also define groups $W_n(A)$ of Witt vectors of length n in A. These are related by restriction maps $R: W_n(A) \to W_{n-1}(A)$ and $W(A)$ is the inverse limit. In particular, $W_1(A)$ is defined to be the quotient of A by the additive subgroup $[A,A]$ generated by elements of the form $xy-yx$, $x,y \in A$. There are natural exact sequences

$$0 \to A/[A,A] \xrightarrow{V^{n-1}} W_n(A) \xrightarrow{R} W_{n-1}(A) \to 0$$

which are useful both for proofs and calculations. We use these in Theorem 1.7.10 below to evaluate $W(A)$ when A is a free associative $\mathbf{F}_p$-algebra without unit. The sequences are usually not split exact, but in contrast to the classical case, this is not even true as functors from rings to sets, i.e. $W_n(A)$ is not naturally bijective to the n-fold product of copies of $A/[A,A]$. Finally, the construction $W(-)$ is Morita invariant:

$$W(M_n(A)) \cong W(A).$$

This more general algebraic structure arises naturally in topology: the topological cyclic homology defined by Bökstedt–Hsiang–Madsen, [BHM], associates to any ring A a (-2)-connected spectrum $\mathrm{TC}(A;p)$. We write $\mathrm{TC}_*(A;p) = \pi_* \mathrm{TC}(A;p)$. In §2 below we prove

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