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A Littlewood–Richardson rule for the K-theory of Grassmannians

by

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1. Introduction

Let $X = \mathrm{Gr}(d, \mathbf{C}^n)$ be the Grassmann variety of d-dimensional subspaces of $\mathbf{C}^n$. The goal of this paper is to give an explicit combinatorial description of the Grothendieck ring $K^\circ X$ of algebraic vector bundles on X.

K-theory of Grassmannians is a special case of K-theory of flag varieties, which was studied by Kostant and Kumar [13] and by Demazure [5]. Lascoux and Schützenberger defined Grothendieck polynomials which give formulas for the structure sheaves of the Schubert varieties in a flag variety [16], [14]. The combinatorial understanding of these polynomials was further developed by Fomin and Kirillov [9], [8].

Recall that if $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_d)$ is a partition with d parts and $\lambda_1 \leqslant n-d$, then the Schubert variety in X associated to λ is the subset

$$\Omega_\lambda = \{V \in \mathrm{Gr}(d, \mathbf{C}^n) \mid \dim(V \cap \mathbf{C}^{n-d+i-\lambda_i}) \geqslant i \text{ for all } 1 \leqslant i \leqslant d\}. \tag{1.1}$$

Here $\mathbf{C}^k \subset \mathbf{C}^n$ denotes the subset of vectors whose last $n-k$ components are zero. The codimension of Ω_λ is equal to the weight $|\lambda| = \sum \lambda_i$ of λ. If we identify partitions with their Young diagrams, then a Schubert variety Ω_μ is contained in Ω_λ if and only if μ contains λ. From the fact that the open Schubert cells $\Omega_\lambda^\circ = \Omega_\lambda \setminus \bigcup_{\mu \supsetneq \lambda} \Omega_\mu$ form a cell decomposition of X, one can deduce that the classes of the structure sheaves $\mathcal{O}_{\Omega_\lambda}$ form a basis for the Grothendieck ring of X.

We will study the structure constants for $K^\circ X$ with respect to this basis. These are the unique integers $c_{\lambda\mu}^\nu$ such that

$$[\mathcal{O}_{\Omega_\lambda}] \cdot [\mathcal{O}_{\Omega_\mu}] = \sum_\nu c_{\lambda\mu}^\nu [\mathcal{O}_{\Omega_\nu}]. \tag{1.2}$$

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