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Local connectivity of some Julia sets containing a circle with an irrational rotation

by

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The Fatou set F_R for a rational map $R: \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is the set of points $z \in \overline{\mathbf{C}}$ possessing a neighbourhood on which the family of iterates $\{R^n\}_{n \geqslant 0}$ is normal (in the sense of Montel). The Julia set $J_R = \overline{\mathbf{C}} - F_R$ is the complement of the Fatou set. (The monographs [CG], [Be], [St] provide introductions to the theory of iteration of rational maps.)

Let $\theta \in]0,1[- \mathbf{Q}$ be an irrational number and write it as a continued fraction

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}}}$$

where $a_n \in \mathbf{N}$ for each $n \geqslant 1$. The number θ is termed of constant type, or equivalently, is termed Diophantine of exponent 2, if the sequence $\{a_n\}_{n \in \mathbf{N}}$ is bounded.

For $\theta \in [0,1]$ define $\lambda_\theta = \exp(i2\pi\theta)$ and $P_\theta(z) := \lambda_\theta z + z^2$. Moreover, let J_{P_θ} denote the Julia set of P_θ. The polynomial P_θ has a Siegel disc around the (indifferent) fixed point 0, if and only if it is locally linearizable. That is, if there exists a local change of coordinates $\phi: (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ with $\phi \circ P_\theta = \lambda_\theta \cdot \phi$. It is well known that P_θ has a Siegel disc around 0 for every θ of constant type (see e.g. [Si]).

THEOREM A. *For every θ of constant type the Julia set J_{P_θ} is locally connected and has zero Lebesgue measure.*

The proof uses in an essential way a model J_θ of J_{P_θ}. The model J_θ was constructed in 1986 and proved to be quasi-conformally equivalent to J_{P_θ} in 1987 (see [Do] for the