Acta Math., 187 (2001), 237–273
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Values of Brownian intersection exponents, I: Half-plane exponents

by

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1. Introduction

Theoretical physics predicts that conformal invariance plays a crucial role in the macroscopic behavior of a wide class of two-dimensional models in statistical physics (see, e.g., [4], [6]). For instance, by making the assumption that critical planar percolation behaves in a conformally invariant way in the scaling limit, and using ideas involving conformal field theory, Cardy [7] produced an exact formula for the limit, as $N\to\infty$, of the probability that, in two-dimensional critical percolation, there exists a cluster crossing the rectangle $[0,aN]\times[0,bN]$. Also, Duplantier and Saleur [13] predicted the "fractal dimension" of the hull of a very large percolation cluster. These are just two examples among many such predictions.

In 1988, Duplantier and Kwon [12] suggested that the ideas of conformal field theory can also be applied to predict the intersection exponents between random walks in $\mathbf{Z}^2$ (and Brownian motions in $\mathbf{R}^2$). They predicted, for instance, that if B and B' are independent planar Brownian motions (or simple random walks in $\mathbf{Z}^2$) started from distinct points in the upper half-plane $\mathbf{H}=\{(x,y):y>0\}=\{z\in\mathbf{C}:\operatorname{Im}(z)>0\}$, then when $n\to\infty$,

$$\mathbf{P}[B[0,n]\cap B'[0,n]=\varnothing]=n^{-\zeta+o(1)} \tag{1.1}$$

and

$$\mathbf{P}[B[0,n]\cap B'[0,n]=\varnothing \text{ and } B[0,n]\cup B'[0,n]\subset\mathbf{H}]=n^{-\tilde{\zeta}+o(1)}, \tag{1.2}$$

where

$$\zeta=\tfrac{5}{8},\quad \tilde{\zeta}=\tfrac{5}{3}.$$