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Holomorphic representation theory II

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Introduction

The starting point in the theory of holomorphic extensions of unitary representations was Ol'shanskiĭ's observation that, if W is a pointed generating invariant cone in a simple Lie algebra $\mathfrak{g}$, G a corresponding linear connected group, and $G_{\mathbf{C}}$ its universal complexification, then the set $S_W = G\exp(iW)$ is a closed subsemigroup of $G_{\mathbf{C}}$ ([O]). This theorem has been generalized by Hilgert and Ólafsson to solvable groups ([HO]) and the most general result of this type, due to Lawson ([La]), is that if $G_{\mathbf{C}}$ is a complex Lie group with an antiholomorphic involution inducing the complex conjugation on $\mathfrak{g}_{\mathbf{C}} = \mathbf{L}(G_{\mathbf{C}})$, then the set $S_W = G\exp(iW)$ is a closed subsemigroup of $G_{\mathbf{C}}$. The class of semigroups obtained by this construction is not sufficient for many applications in representation theory. For instance Howe's oscillator semigroup (cf. [How]) is a 2-fold covering of such a semigroup, but it does not fit into any group. In [Ne6] we have shown that given a Lie algebra $\mathfrak{g}$, a generating invariant convex cone $W \subseteq \mathfrak{g}$, and a discrete central subgroup of the simply connected group corresponding to the Lie algebra $\mathfrak{g} + i(W \cap (-W))$ which is invariant under complex conjugation, there exists a semigroup $S = \Gamma(\mathfrak{g}, W, D)$ called the *Ol'shanskiĭ semigroup* defined by this data. This semigroup is the quotient $\widetilde{S}/D$, where $\widetilde{S}$ is the universal covering semigroup of S (cf. [Ne3]) and $D \cong \pi_1(S)$ is a discrete central subgroup of $\widetilde{S}$. Moreover, the semigroup $\widetilde{S}$, also denoted $\Gamma(\mathfrak{g}, W)$ can be obtained as the universal covering semigroup of the subsemigroup $\langle \exp(\mathfrak{g} + iW) \rangle$ of the simply connected complex Lie group $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{g}_{\mathbf{C}}$.

A *holomorphic representation* of a complex Ol'shanskiĭ semigroup S is a weakly continuous monoid morphism $\pi: S \to B(\mathcal{H})$ into the algebra of bounded operators on a Hilbert space $\mathcal{H}$ such that π is holomorphic on the interior $\operatorname{int}(S)$ of S and π is *involutive*, i.e., $\pi(s^*) = \pi(s)^*$ holds for all $s \in S$. This set is a dense semigroup ideal which is a complex manifold. One can think of representations of S as analytic continuations of unitary representations of the subgroup $U(S) = \{s \in S : s^* s = \mathbf{1}\}$ of unitary elements in S.