Polynomial growth estimates for multilinear singular integral operators

by

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This paper originated with a question of Yves Meyer: Let T be a convolution Calderón-Zygmund operator on $\mathbf{R}^d$, $d\geq 2$, with kernel $K(x-y)$. Let $b^1, \ldots, b^m$ be m L^∞ functions and let $F\in C^\infty(\mathbf{C}^m)$. Does the kernel

$$L(x,y) = K(x-y)\,F\Big(\ldots, \int_0^1 b_i(tx+(1-t)\,y)\,dt, \ldots\Big)$$

define an operator bounded on $L^2(\mathbf{R}^d)$? When $d=1$ this is equivalent to asking whether the nth Calderón commutator is bounded on L^2 with polynomial growth of the operator norm, that is, with a bound Cn^M as $n\to\infty$ [2]. The argument in [1] also reduces the higher-dimensional problem to proving the boundedness of a sequence of operators, which we call the d-commutators, with polynomial growth. The kernel of the nth d-commutator is

$$L(x,y) = K(x-y)\left[\int_0^1 a(tx+(1-t)\,y)\,dt\right]^n$$

where $a\in L^\infty(\mathbf{R}^{\mathrm{d}})$ is complex-valued, and the question is whether

$$\left\|\int L(x,y)f(y)\,dy\right\|_2 \leq Cn^M\|a\|_\infty^n\|f\|_2$$

for all $f\in L^2$, $a\in L^\infty$ and $n\in\mathbf{Z}^+$, the integral being suitably interpreted. This question is motivated in part by work of Leichtnam [3], and in part by the formal analogy with the Calderón commutators. We answer it in the affirmative.

For an arbitrary $a\in L^\infty$, the expresssion $\int_0^1 a(tx+(1-t)\,y)\,dt$ is a far less regular function of x, y when $d\geq 2$ than when $d=1$. Consequently the kernels of the d-commutators fail to satisfy the "standard estimates" of Calderón-Zygmund theory, and the general boundedness criterion of [5] does not apply. In fact the d-commutators