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# A converse to the mean value theorem for harmonic functions

by

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## 0. Introduction

Let $U \neq \varnothing$ be a bounded domain in $\mathbf{R}^d$, $d \geqslant 1$, and for every $x \in U$ let $B^x$ be an open ball contained in $U$ with center $x$. If $f$ is a harmonic integrable function on $U$ then

$$f(x) = \frac{1}{\lambda(B^x)} \int_{B^x} f \, d\lambda \quad \text{for every } x \in U \tag{$*$}$$

($\lambda$ Lebesgue measure on $\mathbf{R}^d$). The converse question to what extent this restricted mean value property implies harmonicity has a long history (we are indebted to I. Netuka for valuable hints). Volterra [26] and Kellogg [20] noted first that a continuous function $f$ on the closure $\overline{U}$ of $U$ satisfying $(*)$ is harmonic on $U$. At least if $U$ is regular there is a very elementary proof for this fact (see Burckel [7]): Let $g$ be the difference between $f$ and the solution of the Dirichlet problem with boundary value $f$. If $g \neq 0$, say $a = \sup g(U) > 0$, choose $x \in \{g = a\}$ having minimal distance to the boundary. Then $(*)$ leads to an immediate contradiction. In fact, for continuous functions on $\overline{U}$ the question is settled for arbitrary harmonic spaces and arbitrary representing measures $\mu_x \neq \varepsilon_x$ for harmonic functions.

If $f$ is bounded on $U$ and Borel measurable the answer may be negative unless restrictions on the radius $r(x)$ of the balls $B^x$ are imposed (Veech [23]): Let $U = ]-1, 1[$, $f(0) = 0$, $f = -1$ on $]-1, 0[$, $f = 1$ on $]0, 1[$, $0 \notin B^x$ for $x \neq 0$ (similarly in $\mathbf{R}^d$, $d \geqslant 2$)!

There are various positive results, sometimes under restrictions on $U$, but always under restrictions on the function $x \mapsto r(x)$ (Feller [9], Akcoglu and Sharpe [1], Baxter [2] and [3], Heath [17], Veech [23] and [24]). For example Heath [17] showed for arbitrary $U$ that a bounded Lebesgue measurable function on $U$ having the restricted mean value property $(*)$ is harmonic provided that, for some $\varepsilon > 0$, $\varepsilon d(x, \complement U) < r(x) < (1-\varepsilon) d(x, \complement U)$ holds for every $x \in U$. Veech [23] proved that a Lebesgue measurable function $f$ on $U$