HYPERGEOMETRIC FUNCTIONS

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Introduction

1. In this paper we shall consider hypergeometric series of the form

$$F\left(\begin{matrix}\alpha_1\ \alpha_2 \dots \alpha_n \\ \gamma_1\ \gamma_2 \dots \gamma_{n-1}\end{matrix}\middle| z\right) = \sum_{\nu=0}^{\infty} \frac{(\alpha_1)_\nu (\alpha_2)_\nu \dots (\alpha_n)_\nu}{\nu!\,(\gamma_1)_\nu \dots (\gamma_{n-1})_\nu} z^\nu, \tag{1}$$

where

$$(\alpha)_\nu = \alpha(\alpha+1)\dots(\alpha+\nu-1), \qquad (\alpha)_0 = 1.$$

When the argument z is omitted, it will be assumed that $z=1$. If $n=2$, it is usual to write

$$F(a, b, c; z) = \sum_{\nu=0}^{\infty} \frac{(a)_\nu (b)_\nu}{\nu!\,(c)_\nu} z^\nu. \tag{2}$$

This particular hypergeometric function has played an important role in the development of analysis due to the classical works of Euler, Gauss, Riemann and Kummer. The general case, where n is an arbitrary integer >2, has first been considered by Thomae [60], who showed that the series (1) satisfy a linear differential equation of the order n, which in the vicinity of $z=0$ has a fundamental system of solutions, represented by hypergeometric series multiplied by a power of z, provided none of the differences between the numbers $0, \gamma_1, \gamma_2 \dots \gamma_{n-1}$ is an integer. Goursat [15] has shown how the definition of the hypergeometric functions by means of their analytic properties, given by Riemann in the case $n=2$, may be extended to functions of an arbitrary order. Furthermore Goursat has considered multiple integrals of the order $n-1$, which represent the hypergeometric functions of the order n, when the parameters satisfy certain conditions. Similar multiple integrals have been used by Poch-