THE BLOSSOMING OF SCHRÖDER'S FOURTH PROBLEM

BY

JOHN RIORDAN

The Rockefeller University New York, N. Y. 10021, USA

1. Introduction

Schröder's fourth problem is the last of four enumeration problems associated with various kinds of bracketing of a sum (product) with a fixed number of terms, considered in [12] and published in 1870. In contrast with the earlier (1838) work of E. Catalan, in which the bracketing was restricted to two neighboring terms, Schröder allowed bracketing of any number of terms and in the fourth problem included the effects of term reordering, in a way which will be clearer in the detailed description in the next section.

The identity of this problem with the enumeration of fully labeled (essentially series) series-parallel arrangements has been noticed in [4], a joint paper with L. Carlitz. However the mapping of the bracketings to series-parallel arrangements was not pursued.

More recently, Louis Comtet, [7], has given a mapping of Schroder's bracketings to trees, described as arborescences bifurcante. It is somewhat surprising that these trees, without labels, also appear in A. Cayley's landmark paper [5], of 1857, devoted mainly to the enumeration of unlabeled rooted trees. They appear as a kind of simplification of the main result and are described mainly by the phrase "every branching is at least a bifurcation". The fact that the enumeration is by number of endpoints, rather than total number of points, as in the main result, is not emphasized. In the terminology of Frank Harary and Geert Prins [8, p. 150], these trees are homeomorphically irreducible planted trees, that is, without points of degree two; I prefer the shorter term *series-reduced*.

The object of this paper, in the first place, is to give the mapping of series-parallel arrangements to both the bracketings of Schröder's fourth problem (which Comtet [7] calls schröderiens, for brevity) and to the Cayley–Comtet trees. The mapping is so simple as to arouse the hope that it may remove the stigma from which series-parallel arrangements seem to have suffered in mathematical circles because of their origin in electric circuit

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