ON NON-LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER: III. THE EQUATION $\ddot{y} - k(1 - y^2)\dot{y} + y = b\mu k \cos(\mu t + \alpha)$ FOR LARGE k, AND ITS GENERALIZATIONS

BY

J. E. LITTLEWOOD

in Cambridge

Introduction[1]

1. We are concerned with equations in real variables of the form

$$\ddot{y} + f(y)\dot{y} + g(y) = p(t),$$

where f, g, p are smooth functions of their arguments, and p has period $\lambda = 2\pi/\mu$ in t. About f we suppose that $\underline{\lim}\, f > 0$ as $y \to \pm\infty$; that is to say, we suppose the "damping" to be positive for large $|y|$. About g we suppose that it has a "restoring" effect, i.e. has the sign of y. The simplest case, and a specially important one, to be covered in any generalization, is $g = ay$ for positive a. We do in fact assume always that $g(0) = 0$, and that g' exists and has a positive lower bound.

There is some general theory of such equations. A trajectory (or "motion") with initial conditions $y(t_0) = \xi$, $\dot{y}(t_0) = \eta$ (ξ, η real) at some fixed $t = t_0$ is said to have the point $P = (\xi, \eta)$ as "representative point". If ξ', η' are the values of y, $\dot{y}$ at $t = t_0 + \lambda$ the transformation T from P to $P' = (\xi', \eta') = TP = T(\xi, \eta)$ is $1 - 1$ and continuous.

With the condition $\underline{\lim}\, f > 0$ and suitable conditions on g (fulfilled for $g = y$), every trajectory is bounded as $t \to \infty$, and T transforms a suitable large domain in the P space into a domain contained in the original one. Further, the vector V, or $P \to TP$, makes exactly one revolution as P moves positively round the boundary. Then a "fixed point" theorem holds, and the "index number" proof of it is valid.[2]

[1] This paper is based on joint work with M. L. CARTWRIGHT.

A paper I, with the same general title, was published in the *Journal London Math. Soc.*, 20 (1945), 180–189, jointly with M. L. CARTWRIGHT. This was written with the same aims as the present Introduction, but in drastically condensed form. We have borrowed some passages from it.

[2] N. LEVINSON, *Journal of Math. and Physics*, 22 (1943), 41–48, and *Annals of Math.*, 45 (1945), 723–727.